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# INHOMOGENEOUS BOUNDARY VALUE PROBLEMS FOR COMPRESSIBLE NAVIER-STOKES EQUATIONS: WELL-POSEDNESS AND SENSITIVITY ANALYSIS

P. I. PLOTNIKOV, E.V. RUBAN AND J. SOKOLOWSKI

**ABSTRACT.** In the paper compressible, stationary Navier-Stokes equations are considered. A framework for analysis of such equations is established. In particular, the well-posedness for inhomogeneous boundary value problems of elliptic-hyperbolic type is shown. Analysis is performed for small perturbations of the so-called *approximate solutions*, i.e., the solutions take form (1.12). The approximate solutions are determined from Stokes problem (1.11). The small perturbations are given by solutions to (1.13). The uniqueness of solutions for problem (1.13) is proved, and in addition, the differentiability of solutions with respect to the coefficients of differential operators is shown. The results on the well-posedness of nonlinear problem are interesting on its own, and are used to obtain the shape differentiability of the drag functional for incompressible Navier-Stokes equations. The shape gradient of the drag functional is derived in the classical and useful for computations form, an appropriate adjoint state is introduced to this end. The shape derivatives of solutions to the Navier-Stokes equations are given by smooth functions, however the shape differentiability is shown in a weak norm. The method of analysis proposed in the paper is general, and can be used to establish the well-posedness for distributed and boundary control problems as well as for inverse problems in the case of the state equations in the form of compressible Navier-Stokes equations. The differentiability of solutions to the Navier-Stokes equations with respect to the data leads to the first order necessary conditions for a broad class of optimization problems.

## 1. INTRODUCTION

Shape optimization for compressible Navier-Stokes equations (NSE) is important for applications [24] and it is investigated from numerical point of view, however the mathematical analysis of such problems is not available in the existing literature. One of the reasons is the lack of the existence results for inhomogeneous boundary value problems for such equations.

The results established in the paper lead in particular to the first order optimality conditions for a class of shape optimization problems for compressible Navier-Stokes equations.

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*Key words and phrases.* Shape optimization, compressible Navier-Stokes equations, drag minimization, transport equations, shape gradient, shape derivative, approximate solutions, shape sensitivity analysis, necessary optimality conditions.

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**1.1. Problem formulation.** In the paper we prove the well-posedness and present the sensitivity analysis for inhomogeneous boundary value problems for the compressible Navier-Stokes equations. We restrict ourselves to the case of a specific shape optimization problem for stationary motion of viscous compressible non heat-conducting isentropic gas. However, the technique of modelling and analysis presented here is general and can be used for a broad class of optimization problems for nonlinear elliptic-hyperbolic equations. The sensitivity analysis is the necessary step for numerical methods of solution for optimization problems. In general the mathematical analysis of optimization problems includes the following steps, with the mathematical proofs of the required facts,

- existence of solutions,
- uniqueness and optimality conditions,
- numerical method of solution.

The existence of an optimal shape for the drag minimization is shown in [35] under the assumptions compared to the assumptions in the present paper. Here, we present the necessary mathematical tools required for the second step of analysis, i.e., the derivation of an optimality system. In particular, we prove the shape differentiability of solutions to (1.9) as well as of drag functional (1.3) and provide the classical representation of the shape derivatives of integral shape functionals in terms of an appropriate adjoint state.

We consider in details all questions on the existence, uniqueness and shape differentiability of solutions to stationary boundary value problems for compressible Navier-Stokes equations. Such boundary value problems can be regarded as the mathematical models of viscous gas flow around a body tested in the wind tunnel. We assume that the viscous gas occupies the double-connected domain  $\Omega = B \setminus S$ , where  $B \subset \mathbb{R}^3$ , is a hold-all domain with the smooth boundary  $\Sigma = \partial B$ , and  $S \subset B$  is a compact obstacle. Furthermore, we assume that the velocity of the gas coincides with a given vector field  $\mathbf{U} \in C^\infty(\mathbb{R}^3)^3$  on the surface  $\Sigma$ . In this framework, the boundary of the flow domain  $\Omega$  is divided into the three subsets, inlet  $\Sigma_{\text{in}}$ , outgoing set  $\Sigma_{\text{out}}$ , and characteristic set  $\Sigma_0$ , which are defined by the equalities

$$(1.1) \quad \Sigma_{\text{in}} = \{x \in \Sigma : \mathbf{U} \cdot \mathbf{n} < 0\}, \quad \Sigma_{\text{out}} = \{x \in \Sigma : \mathbf{U} \cdot \mathbf{n} > 0\},$$

$\Sigma_0 = \{x \in \partial\Omega : \mathbf{U} \cdot \mathbf{n} = 0\}$ , where  $\mathbf{n}$  stands for the outward normal to  $\partial\Omega = \Sigma \cup \partial S$ . In its turn the compact  $\Gamma = \Sigma_0 \cap \Sigma$  splits the surface  $\Sigma$  into three disjoint parts  $\Sigma = \Sigma_{\text{in}} \cup \Sigma_{\text{out}} \cup \Gamma$ . The problem is to find the velocity field  $\mathbf{u}$  and the gas density  $\varrho$  satisfying the following equations along with the boundary conditions

$$(1.2a) \quad \Delta \mathbf{u} + \lambda \nabla \operatorname{div} \mathbf{u} = R \varrho \mathbf{u} \cdot \nabla \mathbf{u} + \frac{R}{\epsilon^2} \nabla p(\varrho) \text{ in } \Omega,$$

$$(1.2b) \quad \operatorname{div}(\varrho \mathbf{u}) = 0 \text{ in } \Omega,$$

$$(1.2c) \quad \mathbf{u} = \mathbf{U} \text{ on } \Sigma, \quad \mathbf{u} = 0 \text{ on } \partial S,$$

$$(1.2d) \quad \varrho = \varrho_0 \text{ on } \Sigma_{\text{in}},$$

where the pressure  $p = p(\varrho)$  is a smooth, strictly monotone function of the density,  $\epsilon$  is the Mach number,  $R$  is the Reynolds number,  $\lambda$  is the viscosity ratio, and  $\varrho_0$  is a positive constant.

For the derivation of equations (1.2) we refer to [22]. The general theory of compressible Navier-Stokes equations is covered by monographs [9], [21] and [28]. In

particular, the main results on the existence of global weak solutions for stationary problems with the zero velocity boundary conditions were established in [21] and sharpened in [28]. See also [12], [33], [34] for generalizations.

There are numerous papers dealing with the zero velocity boundary value problem to steady compressible Navier-Stokes equations in the context of small data. We recall only that there are three different approaches to this problem proposed in [2], [30], and [25]. The basic results on the local existence and uniqueness of strong solutions are assembled in [28]. For an interesting overview see [31].

The inhomogeneous boundary problems were studied in papers [17]-[18], where the local existence and uniqueness results were obtained in two dimensional case under the assumption that the velocity  $\mathbf{u}$  is close to a given constant vector. The question of the existence of strong solutions to boundary value problems in three spatial dimensions with nonzero velocity boundary data in smooth domains is still an open problem. There are difficulties including the problem of the total mass control and of the singularities developed by solutions at the manifold  $\overline{\Sigma}_{in} \cap \overline{\Sigma}_0 \cup \overline{\Sigma}_{out}$ . In the paper we prove the local existence and uniqueness of strong solutions to problem (1.2) in fractional Sobolev spaces, under the assumption that the given vector field  $\mathbf{U}$  satisfies the emergent vector field conditions **(H1)**-**(H3)** on  $\Gamma$ . It seems that a condition of this type is necessary for the continuity of mass density  $\varrho$ .

*Shape optimization problems.* Among many shape optimization problems for Navier-Stokes equations we could list the drag minimization, which is investigated in this paper and in [32]-[35]. Another problem of practical interest concerns optimal shape of tunnels [24]. In the specific problem the required mass distribution on the outlet of the tunnel is given. The associated shape optimization problem can be formulated as follows. Determine an admissible domain such that the mass distribution at the inlet is given, the velocity field is prescribed on the boundary of the domain, and the mass distribution at the outlet is as close as possible to a given function. Inlet and outlet subsets are defined by the vector field  $\mathbf{U}$  which serves as the inhomogeneous boundary condition for the law of momentum conservation in the form of Navier-Stokes stationary system. The shape optimization problem as it is formulated in [24], enters in our framework, and the results on shape sensitivity analysis can be applied to solve the problem. Another class of problems which can be investigated using the tools proposed in the paper are optimal control problems, e.g. with the boundary controls. These subjects are however beyond the scope of the paper, and we present as an example to the general theory the drag minimization problem.

*Drag minimization.* One of the main applications of the theory of compressible viscous flows is the optimal shape design in aerodynamics. The classical sample is the problem of the minimization of the drag of airfoil travelling in atmosphere with uniform speed  $\mathbf{U}_\infty$ . Recall that in our framework the hydro-dynamical force acting on the body  $S$  is defined by the formula [36],

$$\mathbf{J}(S) = - \int_{\partial S} (\nabla \mathbf{u} + (\nabla \mathbf{u})^* + (\lambda - 1) \operatorname{div} \mathbf{u} \mathbf{I} - \frac{R}{\epsilon^2} p \mathbf{I}) \cdot \mathbf{n} dS .$$

In a frame attached to the moving body the drag is the component of  $\mathbf{J}$  parallel to  $\mathbf{U}_\infty$ ,

$$(1.3) \quad J_D(S) = \mathbf{U}_\infty \cdot \mathbf{J}(S),$$

and the lift is the component of  $\mathbf{J}$  in the direction orthogonal to  $\mathbf{U}_\infty$ . For the fixed data, the drag can be regarded as a functional depending on the shape of the obstacle  $S$ . The minimization of the drag and the maximization of the lift are between shape optimization problems of some practical importance. The questions of the domain dependence of solutions to non-stationary compressible NSE and on the solvability of the drag optimization problem were considered in papers [10],[11]. The solvability of the drag minimization problem for stationary equations (1.2) is shown in [32], [35]. For incompressible Navier-Stokes equations, the existence of shape derivatives of solutions and the formulae for the shape derivative of the drag functional and adjoint state were obtained in [4], [5] and [37], see also [38] and [39] for some generalizations. The growing literature on numerical and applied aspects of the problem is nicely surveyed in [15] and [24]. To our best knowledge, the mathematical sensitivity analysis for the compressible Navier-Stokes equations has not been studied yet. We derive the formulae for the shape derivatives of the drag functional which can be used, in particular, for the explicit formulation of optimality conditions. In order to define the shape derivatives of the shape functional we combine the shape derivatives of the solutions to the governing PDE's with an appropriate adjoint state according to the same scheme as it is proposed e.g., in [37] for steady incompressible equations.

We start with description of our framework for shape sensitivity analysis, or more general, for well-posedness of compressible NSE. To this end we choose the vector field  $\mathbf{T} \in C^2(\mathbb{R}^3)^3$  vanishing in the vicinity of  $\Sigma$ , and define the mapping

$$(1.4) \quad y = x + \varepsilon \mathbf{T}(x),$$

which describes the perturbation of the shape of the obstacle. We refer the reader to [40] for more general framework and results in shape optimization. For small  $\varepsilon$ , the mapping  $x \rightarrow y$  takes diffeomorphically the flow region  $\Omega$  onto  $\Omega_\varepsilon = B \setminus S_\varepsilon$ , where the perturbed obstacle  $S_\varepsilon = y(S)$ . Let  $(\bar{\mathbf{u}}_\varepsilon, \bar{\varrho}_\varepsilon)$  be solutions to problem (1.2) in  $\Omega_\varepsilon$ . After substituting  $(\bar{\mathbf{u}}_\varepsilon, \bar{\varrho}_\varepsilon)$  into the formulae for  $\mathbf{J}$ , the drag becomes the function of the parameter  $\varepsilon$ . Our aim is, in fact, to prove that this function is well-defined and differentiable at  $\varepsilon = 0$ . This leads to the first order shape sensitivity analysis for solutions to compressible Navier-Stokes equations. It is convenient to reduce such an analysis to the analysis of dependence of solutions with respect to the coefficients of the governing equations. To this end, we introduce the functions  $\mathbf{u}_\varepsilon(x)$  and  $\varrho_\varepsilon(x)$  defined in the unperturbed domain  $\Omega$  by the formulae

$$\mathbf{u}_\varepsilon(x) = \mathbf{N} \bar{\mathbf{u}}_\varepsilon(x + \varepsilon \mathbf{T}(x)), \quad \varrho_\varepsilon(x) = \bar{\varrho}_\varepsilon(x + \varepsilon \mathbf{T}(x)),$$

where

$$(1.5) \quad \mathbf{N}(x) = \det(\mathbf{I} + \varepsilon \mathbf{T}'(x))(\mathbf{I} + \varepsilon \mathbf{T}'(x))^{-1}.$$

is the adjugate matrix of the Jacobi matrix  $\mathbf{I} + \varepsilon \mathbf{T}'$ . Furthermore, we also use the notation  $\mathbf{g}(x) = \sqrt{\det \mathbf{N}}$ . It is easily to see that the matrices  $\mathbf{N}(x)$  depends analytically upon the small parameter  $\varepsilon$  and

$$(1.6) \quad \mathbf{N} = \mathbf{I} + \varepsilon \mathbf{D}(x) + \varepsilon^2 \mathbf{D}_1(\varepsilon, x),$$

where  $\mathbf{D} = \operatorname{div} \mathbf{T}\mathbf{I} - \mathbf{T}'$ . Calculations show that for  $\mathbf{u}_\varepsilon, \varrho_\varepsilon$ , the following boundary value problem is obtained

$$(1.7a) \quad \Delta \mathbf{u}_\varepsilon + \nabla \left( \lambda \mathbf{g}^{-1} \operatorname{div} \mathbf{u}_\varepsilon - \frac{R}{\varepsilon^2} p(\varrho_\varepsilon) \right) = \mathcal{A} \mathbf{u}_\varepsilon + R \mathcal{B}(\varrho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) \text{ in } \Omega,$$

$$(1.7b) \quad \operatorname{div} (\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0 \text{ in } \Omega,$$

$$(1.7c) \quad \mathbf{u}_\varepsilon = \mathbf{U} \text{ on } \Sigma, \quad \mathbf{u}_\varepsilon = 0 \text{ on } \partial S,$$

$$(1.7d) \quad \varrho_\varepsilon = \varrho_0 \text{ on } \Sigma_{\text{in}}.$$

Here, the linear operator  $\mathcal{A}$  and the nonlinear mapping  $\mathcal{B}$  are defined in terms of  $\mathbf{N}$ ,

$$(1.8) \quad \begin{aligned} \mathcal{A}(\mathbf{u}) &= \Delta \mathbf{u} - \mathbf{N}^{-1} \operatorname{div} (\mathbf{g}^{-1} \mathbf{N} \mathbf{N}^* \nabla (\mathbf{N}^{-1} \mathbf{u})), \\ \mathcal{B}(\varrho, \mathbf{u}, \mathbf{w}) &= \varrho (\mathbf{N}^*)^{-1} \left( \mathbf{u} \nabla (\mathbf{N}^{-1} \mathbf{w}) \right). \end{aligned}$$

The specific structure of the matrix  $\mathbf{N}$  does not play any particular role in the further analysis. Therefore, we consider a general problem of the existence, uniqueness and dependence on coefficients of the solutions to equations (1.7) under the assumption that  $\mathbf{N}$  is a given matrix-valued function which is close, in an appropriate norm, to the identity mapping  $\mathbf{I}$  and coincides with  $\mathbf{I}$  in the vicinity of  $\Sigma$ . By abuse of notations, we write simply  $\mathbf{u}$  and  $\varrho$  instead of  $\mathbf{u}_\varepsilon$  and  $\varrho_\varepsilon$ , when studying the well-posedness and dependence on  $\mathbf{N}$ . Before formulation of main results we write the governing equation in more transparent form using the change of unknown functions proposed in [30]. To do so we introduce *the effective viscous pressure*

$$q = \frac{R}{\varepsilon^2} p(\varrho) - \lambda \mathbf{g}^{-1} \operatorname{div} \mathbf{u},$$

and rewrite equations (1.7) in the equivalent form

$$(1.9a) \quad \Delta \mathbf{u} - \nabla q = \mathcal{A}(\mathbf{u}) + R \mathcal{B}(\varrho, \mathbf{u}, \mathbf{u}) \text{ in } \Omega,$$

$$(1.9b) \quad \operatorname{div} \mathbf{u} = a \sigma_0 p(\varrho) - \frac{\mathbf{g} q}{\lambda} \text{ in } \Omega,$$

$$(1.9c) \quad \mathbf{u} \cdot \nabla \varrho + \mathbf{g} \sigma_0 p(\varrho) \varrho = \frac{\mathbf{g} q}{\lambda} \varrho \text{ in } \Omega,$$

$$(1.9d) \quad \mathbf{u} = \mathbf{U} \text{ on } \Sigma, \quad \mathbf{u} = 0 \text{ on } \partial S,$$

$$(1.9e) \quad \varrho = \varrho_0 \text{ on } \Sigma_{\text{in}}.$$

where  $\sigma_0 = R/(\lambda \varepsilon^2)$ . In the new variables  $(\mathbf{u}, q, \varrho)$  the expression for the force  $\mathbf{J}$  reads

$$(1.10) \quad \mathbf{J} = - \int_{\Omega} [\mathbf{g}^{-1} (\mathbf{N}^* \nabla (\mathbf{N} \mathbf{u}) + \nabla (\mathbf{N} \mathbf{u})^* \mathbf{N} - \operatorname{div} \mathbf{u}) - q - R \varrho \mathbf{u} \otimes \mathbf{u}] \mathbf{N}^* \nabla \eta \, dx.$$

where  $\eta \in C^\infty(\Omega)$  is an arbitrary function, which is equal to 1 in an open neighborhood of the obstacle  $S$  and 0 in a vicinity of  $\Sigma$ . The value of  $\mathbf{J}$  is independent of the choice of the function  $\eta$ .

We assume that  $\lambda \gg 1$  and  $R \ll 1$ , which corresponds to almost incompressible flow with low Reynolds number. In such a case, the *approximate solutions* to problem (1.9) can be chosen in the form  $(\varrho_0, \mathbf{u}_0, q_0)$ , where  $\varrho_0$  is a constant in boundary condition (1.9e), and  $(\mathbf{u}_0, q_0)$  is a solution to the boundary value problem for the

Stokes equations,

$$(1.11) \quad \begin{aligned} \Delta \mathbf{u}_0 - \nabla q_0 &= 0, \quad \operatorname{div} \mathbf{u}_0 = 0 \text{ in } \Omega, \\ \mathbf{u}_0 &= \mathbf{U} \text{ on } \Sigma, \quad \mathbf{u}_0 = 0 \text{ on } \partial S, \quad \Pi q_0 = q_0. \end{aligned}$$

In our notations  $\Pi$  is the projector,

$$\Pi u = u - \frac{1}{\operatorname{meas} \Omega} \int_{\Omega} u \, dx.$$

Equations (1.11) can be obtained as the limit of equations (1.9) for the passage  $\lambda \rightarrow \infty$ ,  $R \rightarrow 0$ . It follows from the standard elliptic theory that for the boundary  $\partial\Omega \in C^\infty$ , we have  $(\mathbf{u}_0, q_0) \in C^\infty(\Omega)$ . We look for solutions to problem (1.9) in the form

$$(1.12) \quad \mathbf{u} = \mathbf{u}_0 + \mathbf{v}, \quad \varrho = \varrho_0 + \varphi, \quad q = q_0 + \lambda \sigma_0 p(\varrho_0) + \pi + \lambda m,$$

with the unknowns functions  $\vartheta = (\mathbf{v}, \pi, \varphi)$  and the unknown constant  $m$ . Substituting (1.12) into (1.9) we obtain the following boundary problem for  $\vartheta$ ,

$$(1.13a) \quad \begin{aligned} \Delta \mathbf{v} - \nabla \pi &= \mathcal{A}(\mathbf{u}) + R\mathcal{B}(\varrho, \mathbf{u}, \mathbf{u}) \text{ in } \Omega, \\ \operatorname{div} \mathbf{v} &= \mathbf{g} \left( \frac{\sigma}{\varrho_0} \varphi - \Psi[\vartheta] - m \right) \text{ in } \Omega, \\ \mathbf{u} \cdot \nabla \varphi + \sigma \varphi &= \Psi_1[\vartheta] + m \mathbf{g} \varrho \text{ in } \Omega, \\ \mathbf{v} &= 0 \text{ on } \partial\Omega, \quad \varphi = 0 \text{ on } \Sigma_{\text{in}}, \quad \Pi \pi = \pi, \end{aligned}$$

where

$$\begin{aligned} \Psi_1[\vartheta] &= \mathbf{g} \left( \varrho \Psi[\vartheta] - \frac{\sigma}{\varrho_0} \varphi^2 \right) + \sigma \varphi (1 - \mathbf{g}), \quad \Psi[\vartheta] = \frac{q_0 + \pi}{\lambda} - \frac{\sigma}{p'(\varrho_0) \varrho_0} H(\varphi), \\ \sigma &= \sigma_0 p'(\varrho_0) \varrho_0, \quad H(\varphi) = p(\varrho_0 + \varphi) - p(\varrho_0) - p'(\varrho_0) \varphi, \end{aligned}$$

the vector field  $\mathbf{u}$  and the function  $\varrho$  are given by (1.12). Finally, we specify the constant  $m$ . In our framework, in contrast to the case of homogeneous boundary problem, the solution to such a problem is not trivial. Note that, since  $\operatorname{div} \mathbf{v}$  is of the null mean value, the right-hand side of equation (1.13a)<sub>3</sub> must satisfy the compatibility condition

$$m \int_{\Omega} \mathbf{g} \, dx = \int_{\Omega} \mathbf{g} \left( \frac{\sigma}{\varrho_0} \varphi - \Psi[\vartheta] \right) dx,$$

which formally determines  $m$ . This choice of  $m$  leads to essential mathematical difficulties. To make this issue clear note that in the simplest case  $\mathbf{g} = 1$  we have  $m = \varrho_0^{-1} \sigma (\mathbf{I} - \Pi) \varphi + O(|\vartheta|^2, \lambda^{-1})$ , and the principal linear part of the governing equations (1.13a) becomes

$$\begin{pmatrix} \Delta & -\nabla & 0 \\ \operatorname{div} & 0 & -\frac{\sigma}{\varrho_0} \\ 0 & 0 & \mathbf{u} \cdot \nabla + \sigma \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ \pi \\ \varphi \end{pmatrix} + \begin{pmatrix} 0 \\ m \\ -m \varrho_0 \end{pmatrix} \sim \begin{pmatrix} \Delta \mathbf{v} - \nabla \pi \\ \operatorname{div} \mathbf{v} - \frac{\sigma}{\varrho_0} \Pi \varphi \\ \mathbf{u} \cdot \nabla \varphi + \sigma \Pi \varphi \end{pmatrix}$$

Hence, the question of solvability of the linearized equations derived for (1.13) can be reduced to the question of solvability of the boundary value problem for nonlocal transport equation

$$\mathbf{u} \cdot \nabla \varphi + \sigma \Pi \varphi = f,$$

which is very difficult because of the loss of maximum principle. In fact, this question is concerned with the problem of the control of the total gas mass in compressible flows. Recall that the absence of the mass control is the main obstacle for proving the global solvability of inhomogeneous boundary problems for compressible Navier-Stokes equations, we refer to [21] for discussion. In order to cope with this difficulty we write the compatibility condition in a sophisticated form, which allows us to control the total mass of the gas. To this end we introduce the auxiliary function  $\zeta$  satisfying the equations

$$(1.13b) \quad -\operatorname{div}(\mathbf{u}\zeta) + \sigma\zeta = \sigma\mathbf{g} \text{ in } \Omega, \quad \zeta = 0 \text{ on } \Sigma_{\text{out}},$$

and fix the constant  $m$  as follows

$$(1.13c) \quad m = \varkappa \int_{\Omega} (\varrho_0^{-1}\Psi_1[\vartheta]\zeta - \mathbf{g}\Psi[\vartheta]) dx, \quad \varkappa = \left( \int_{\Omega} \mathbf{g}(1 - \zeta - \varrho_0^{-1}\zeta\varphi) dx \right)^{-1}.$$

In this way the auxiliary function  $\zeta$  becomes an integral part of the solution to problem (1.13). Now, our aim is to prove the existence and uniqueness of solutions to problem (1.13) and investigate the dependence of the solutions on matrices  $\mathbf{N}$ . Before the presentation of main results we introduce some notations and formulate preliminary results.

*Geometrical conditions on the flow region.* We assume that a surface  $\Sigma = \Sigma_{\text{in}} \cup \Sigma_{\text{out}} \cup \Gamma$  and a given vector field  $\mathbf{U}$  satisfy the following conditions, referred to as the *emergent vector field conditions*.

- (H1) The boundary of  $\Omega$  belongs to class  $C^{2+\alpha}$ ,  $\alpha \in (0, 1)$ . For each point  $P \in \Gamma$  there exist the local Cartesian coordinates  $(x_1, x_2, x_3)$  with the origin at  $P$  such that in the new coordinates  $\mathbf{U}(P) = (U, 0, 0)$  with  $U = |\mathbf{U}(P)|$ , and  $\mathbf{n}(P) = (0, 0, -1)$ . Moreover, there is a neighborhood  $\mathcal{O} = [-k, k]^2 \times [-t, t]$  of  $P$  such that the intersections  $\Sigma \cap \mathcal{O}$  and  $\Gamma \cap \mathcal{O}$  are defined by the equations

$$F_0(x) \equiv x_3 - F(x_1, x_2) = 0, \quad \nabla F_0(x) \cdot \mathbf{U}(x) = 0,$$

and  $\Omega \cap \mathcal{O}$  is the epigraph  $\{F_0 > 0\} \cap \mathcal{O}$ . The function  $F$  satisfies the conditions

$$(1.14) \quad \|F\|_{C^2([-k, k]^2)} \leq K, \quad F(0, 0) = 0, \quad \nabla F(0, 0) = 0,$$

where the constants  $k, t < 1$  and  $K > 1$  depend only on the curvature of  $\Sigma$  and are independent of the point  $P$ .

- (H2) For a suitable choice of the constant  $k$ , with  $k$  independent of  $P \in \Gamma$ , the manifold  $\Gamma \cap \mathcal{O}$  admits the parameterization

$$(1.15) \quad x = \mathbf{x}^0(x_2) := (\Upsilon(x_2), x_2, F(\Upsilon(x_2), x_2)),$$

such that  $\Upsilon(0) = 0$  and  $\|\Upsilon\|_{C^2([-k, k])} \leq C_{\Gamma}$ , where the constant  $C_{\Gamma} > 1$  depends only on  $\Sigma$  and  $\mathbf{U}$ .

- (H3) There are positive constants  $N^{\pm}$  independent of  $P$  such that for  $x \in \Sigma$  given by the condition  $F_0(x_1, x_2, x_3) = x_3 - F(x_1, x_2) = 0$  we have

$$(1.16) \quad \begin{aligned} N^-(x_1 - \Upsilon(x_2)) &\leq -\nabla F_0(x) \cdot \mathbf{U}(x) \leq N^+(x_1 - \Upsilon(x_2)) \text{ for } x_1 > \Upsilon(x_2), \\ -N^-(x_1 - \Upsilon(x_2)) &\leq \nabla F_0(x) \cdot \mathbf{U}(x) \leq -N^+(x_1 - \Upsilon(x_2)) \text{ for } x_1 < \Upsilon(x_2). \end{aligned}$$

These conditions have simple geometric interpretation, that  $\mathbf{U} \cdot \mathbf{n}$  only vanishes up to the first order at  $\Gamma$ , and  $\mathbf{U}$  is transversal to  $\Gamma$ , furthermore, for each point  $P \in \Gamma$ , the vector  $\mathbf{U}(P)$  points to the part of  $\Sigma$  where  $\mathbf{U}$  is an exterior vector field. In other



words,  $\mathbf{U}$  and  $\Gamma$  satisfy the so-called *emergent vector field* condition which plays an important role in the theory of the classical oblique derivative problem, see [14].

*Function spaces.* In this paragraph we assemble some technical results which are used throughout of the paper. Function spaces play a central role, and we recall some notations, fundamental definitions and properties, which can be found in [1] and [6]. For the convenience of readers we collect the basic facts from the theory of interpolation spaces in Appendix B. For our applications we need the results in three spatial dimensions, however the results are presented for the dimension  $d \geq 2$ .

Let  $\Omega$  be the whole space  $\mathbb{R}^d$  or a bounded domain in  $\mathbb{R}^d$  with the boundary  $\partial\Omega$  of class  $C^1$ . For an integer  $l \geq 0$  and for an exponent  $r \in [1, \infty)$ , we denote by  $H^{l,r}(\Omega)$  the Sobolev space endowed with the norm  $\|u\|_{H^{l,r}(\Omega)} = \sup_{|\alpha| \leq l} \|\partial^\alpha u\|_{L^r(\Omega)}$ . For real  $0 < s < 1$ , the fractional Sobolev space  $H^{s,r}(\Omega)$  is obtained by the interpolation between  $L^r(\Omega)$  and  $H^{1,r}(\Omega)$ , and consists of all measurable functions with the finite norm

$$\|u\|_{H^{s,r}(\Omega)} = \|u\|_{L^r(\Omega)} + |u|_{s,r,\Omega},$$

where

$$(1.17) \quad |u|_{s,r,\Omega}^r = \int_{\Omega \times \Omega} |x - y|^{-d-rs} |u(x) - u(y)|^r dx dy.$$

In the general case, the Sobolev space  $H^{l+s,r}(\Omega)$  is defined as the space of measurable functions with the finite norm  $\|u\|_{H^{l+s,r}(\Omega)} = \sup_{|\alpha| \leq l} \|\partial^\alpha u\|_{H^{s,r}(\Omega)}$ . For  $0 < s < 1$ , the Sobolev space  $H^{s,r}(\Omega)$  is, in fact [6], the interpolation space  $[L^r(\Omega), H^{1,r}(\Omega)]_{s,r}$ .

Furthermore, the notation  $H_0^{l,r}(\Omega)$ , with an integer  $l$ , stands for the closed subspace of the space  $H^{l,r}(\Omega)$  of all functions  $u \in L^r(\Omega)$  which being extended by zero outside of  $\Omega$  belong to  $H^{l,r}(\mathbb{R}^d)$ .

Denote by  $\mathcal{H}_0^{0,r}(\Omega)$  and  $\mathcal{H}_0^{1,r}(\Omega)$  the subspaces of  $L^r(\mathbb{R}^d)$  and  $H^{1,r}(\mathbb{R}^d)$ , respectively, of all functions vanishing outside of  $\Omega$ . Obviously  $\mathcal{H}_0^{1,r}(\Omega)$  and  $H_0^{1,r}(\Omega)$  are isomorphic topologically and algebraically and we can identify them. However, we need the interpolation spaces  $\mathcal{H}_0^{s,r}(\Omega)$  for non-integers, in particular for  $s = 1/r$ .

**Definition 1.1.** . For all  $0 < s \leq 1$  and  $1 < r < \infty$ , we denote by  $\mathcal{H}_0^{s,r}(\Omega)$  the interpolation space  $[\mathcal{H}_0^{0,r}(\Omega), \mathcal{H}_0^{1,r}(\Omega)]_{s,r}$  endowed with the one of the equivalent norms (6.1) or (6.3) defined by interpolation method.

It follows from the definition of interpolation spaces (see Appendix B) that  $\mathcal{H}_0^{s,r}(\Omega) \subset H^{s,r}(\mathbb{R}^d)$  and for all  $u \in \mathcal{H}_0^{s,r}(\Omega)$ ,

$$(1.18) \quad \|u\|_{H^{s,r}(\mathbb{R}^d)} \leq c(r, s) \|u\|_{\mathcal{H}_0^{s,r}(\Omega)}, \quad u = 0 \text{ outside } \Omega.$$

In other words,  $\mathcal{H}_0^{s,r}(\Omega)$  consists of all elements  $u \in H^{s,r}(\Omega)$  such that the extension  $\bar{u}$  of  $u$  by 0 outside of  $\Omega$  have the finite  $[\mathcal{H}_0^{0,r}(\Omega), \mathcal{H}_0^{1,r}(\Omega)]_{s,r}$ - norm. We identify  $u$  and  $\bar{u}$  for the elements  $u \in \mathcal{H}_0^{s,r}(\Omega)$ . With this identification it follows that  $H_0^{1,r}(\Omega) \subset \mathcal{H}_0^{s,r}(\Omega)$  and the space  $C_0^\infty(\Omega)$  is dense in  $\mathcal{H}_0^{s,r}(\Omega)$ . It is worthy to note that for  $0 < s < 1$  and for  $1 < r < \infty$ , the function  $\bar{u}$  belongs to the space  $H^{s,r}(\mathbb{R}^d)$  if and only if  $u \in H^{s,r}(\Omega)$  and  $\text{dist}(x, \partial\Omega)^{-s} u \in L^r(\Omega)$ . We also point out that the interpolation space  $\mathcal{H}_0^{s,r}(\Omega)$  coincides with the Sobolev space  $H_0^{s,r}(\Omega)$  for  $s \neq 1/r$ . Recall that the standard space  $H_0^{s,r}(\Omega)$  is the completion of  $C_0^\infty(\Omega)$  in the  $H^{s,r}(\Omega)$ -norm.

*Embedding theorems.* For  $sr > d$  and  $0 \leq \alpha < s - r/d$ , the embedding  $H^{s,r}(\Omega) \hookrightarrow C^\alpha(\Omega)$  is continuous and compact. In particular, for  $sr > d$ , the Sobolev space  $H^{s,r}(\Omega)$  is a commutative Banach algebra, i.e. for all  $u, v \in H^{s,r}(\Omega)$ ,

$$(1.19) \quad \|uv\|_{H^{s,r}(\Omega)} \leq c(r, s) \|u\|_{H^{s,r}(\Omega)} \|v\|_{H^{s,r}(\Omega)}.$$

If  $sr < d$  and  $t^{-1} = r^{-1} - d^{-1}s$ , then the embedding  $H^{s,r}(\Omega) \hookrightarrow L^t(\Omega)$  is continuous. In particular, for  $\alpha \leq s$ ,  $(s - \alpha)r < d$  and  $\beta^{-1} = r^{-1} - d^{-1}(s - \alpha)$ ,

$$(1.20) \quad \|u\|_{H^{\alpha,\beta}(\Omega)} \leq c(r, s, \alpha, \beta, \Omega) \|u\|_{H^{s,r}(\Omega)}.$$

It follows from (1.18) that all the embedding inequalities remain true for the elements of the interpolation space  $\mathcal{H}_0^{s,r}(\Omega)$ .

*Duality.* We define

$$(1.21) \quad \langle u, v \rangle = \int_{\Omega} u v \, dx$$

for any functions such that the right hand side make sense. For  $r \in (1, \infty)$ , each element  $v \in L^{r'}(\Omega)$ ,  $r' = r/(r-1)$ , determines the functional  $L_v$  of  $(\mathcal{H}_0^{s,r}(\Omega))'$  by the identity  $L_v(u) = \langle u, v \rangle$ . We introduce the  $(-s, r')$ -norm of an element  $v \in L^{r'}(\Omega)$  to be by definition the norm of the functional  $L_v$ , that is

$$(1.22) \quad \|v\|_{\mathcal{H}^{-s,r'}(\Omega)} = \sup_{\substack{u \in \mathcal{H}_0^{s,r}(\Omega) \\ \|u\|_{\mathcal{H}_0^{s,r}(\Omega)} = 1}} |\langle u, v \rangle|.$$

Let  $\mathcal{H}^{-s,r'}(\Omega)$  denote the completion of the space  $L^{r'}(\Omega)$  with respect to  $(-s, r')$ -norm. For an integer  $s$ ,  $\mathcal{H}^{-s,r'}(\Omega)$  is topologically and algebraically isomorphic to  $(H_0^{s,r}(\Omega))'$ . The same conclusion holds true for all  $s \in (0, 1)$ . Moreover, we can identify  $\mathcal{H}^{-s,r'}(\Omega)$  with the interpolation space  $[L^{r'}(\Omega), H_0^{-1,r'}(\Omega)]_{s,r}$ , see [6] and Appendix B. With this denotations we have the duality principle

$$(1.23) \quad \|u\|_{\mathcal{H}_0^{s,r}(\Omega)} = \sup_{\substack{v \in C_0^\infty(\Omega) \\ \|v\|_{\mathcal{H}^{-s,r'}(\Omega)} = 1}} |\langle u, v \rangle|.$$

With applications to the theory of Navier-Stokes equations in mind, we introduce the smaller dual space defined as follows. We identify the function  $v \in L^{r'}(\Omega)$  with the functional  $L_v \in (H^{s,r}(\Omega))'$  and denote by  $\mathbb{H}^{-s,r'}(\Omega)$  the completion of  $L^{r'}(\Omega)$  in the norm

$$(1.24) \quad \|v\|_{\mathbb{H}^{-s,r'}(\Omega)} := \sup_{\substack{u \in H^{s,r}(\Omega) \\ \|u\|_{H^{s,r}(\Omega)} = 1}} |\langle u, v \rangle|.$$

In the sense of this identification the space  $C_0^\infty(\Omega)$  is dense in the interpolation space  $\mathbb{H}^{-s,r}(\Omega)$ . It follows immediately from the definition that

$$\mathbb{H}^{-s,r'}(\Omega) \subset (H^{s,r}(\Omega))' \subset \mathcal{H}^{-s,r'}(\Omega).$$

For an arbitrary bounded domain  $\Omega \subset \mathbb{R}^3$  with a Lipschitz boundary, we introduce the Banach spaces

$$X^{s,r} = H^{s,r}(\Omega) \cap H^{1,2}(\Omega), \quad Y^{s,r} = H^{s+1,r}(\Omega) \cap H^{2,2}(\Omega), \quad Z^{s,r} = \mathcal{H}^{s-1,r}(\Omega) \cap L^2(\Omega)$$

equipped with the norms

$$\begin{aligned}\|u\|_{X^{s,r}} &= \|u\|_{H^{s,r}(\Omega)} + \|u\|_{H^{1,2}(\Omega)}, \quad \|u\|_{Y^{s,r}} = \|u\|_{H^{1+s,r}(\Omega)} + \|u\|_{H^{2,2}(\Omega)}, \\ \|u\|_{Z^{s,r}} &= \|u\|_{\mathcal{H}^{s-1,r}(\Omega)} + \|u\|_{L^2(\Omega)}.\end{aligned}$$

It can be easily seen that the embeddings  $Y^{s,r} \hookrightarrow X^{s,r} \hookrightarrow Z^{s,r}$  are compact and for  $sr > 3$ , each of the spaces  $X^{s,r}$  and  $Y^{s,r}$  is a commutative Banach algebra.

*Stokes equations.* The following lemma is a straightforward consequence of the classical results on solvability first boundary value problem for the Stokes equations (see [7]) and the interpolation theory.

**Lemma 1.2.** *Let  $\Omega \subset \mathbb{R}^3$  be a bounded domain with  $\partial\Omega \in C^2$  and  $(F, G) \in \mathcal{H}^{s-1,r}(\Omega) \times H^{s,r}(\Omega)$  ( $0 \leq s \leq 1, 1 < r < \infty$ ) Then the boundary value problem*

$$(1.25) \quad \begin{aligned}\Delta \mathbf{v} - \nabla \pi &= F, \quad \operatorname{div} \mathbf{v} = \Pi G \quad \text{in } \Omega, \\ \mathbf{u} &= 0 \quad \text{on } \partial\Omega, \quad \Pi \pi = \pi,\end{aligned}$$

*has a unique solution  $(\mathbf{u}, \pi) \in H^{s+1,r}(\Omega) \times H^{s,r}(\Omega)$  such that*

$$(1.26) \quad \|\mathbf{v}\|_{H^{s+1,r}(\Omega)} + \|\pi\|_{H^{s,r}(\Omega)} \leq c(\Omega, r, s)(\|F\|_{\mathcal{H}^{s-1,r}(\Omega)} + \|G\|_{H^{s,r}(\Omega)}).$$

*In particular, we have*

$$\|\mathbf{v}\|_{Y^{s,r}} + \|\pi\|_{X^{s,r}} \leq c(\Omega, r, s)(\|F\|_{Z^{s,r}} + \|G\|_{X^{s,r}}).$$

*Proof.* The proof is in Appendix B. □

**1.2. Results. Transport equations.** The progress in the theory of compressible Navier-Stokes equations strongly depends on the progress in the theory of transport equations, which is an important part of general theory of the second order partial differential equations with nonnegative characteristic forms. By nowadays there exists a complete theory of generalized solutions to the class of hyperbolic-elliptic equations developed in [8] and [29] under the assumptions that the equations have  $C^1$  coefficients and satisfy the maximum principle. The questions on smoothness properties of solutions are more difficult. We recall the classical results of [16], [29], related to the case of  $\bar{\Sigma}_{\text{in}} \cap \bar{\Sigma}_{\text{out}} = \emptyset$ . The particular case, with  $\Sigma_{\text{in}} = \Sigma_{\text{out}} = \emptyset$ , in the Sobolev spaces is covered in the papers [3] and [25], [26]. The case of nonempty interface  $\Gamma = \bar{\Sigma}_{\text{in}} \cap \bar{\Sigma}_{\text{out}}$  is still weakly investigated. In general case the existence of strong solutions to inhomogeneous boundary value problems for transport equations is still an open problem. The following theorem, which is used throughout of the paper, partially fills this gap. Let us consider the following boundary value problems for linear transport equations

$$(1.27) \quad \mathcal{L}\varphi := \mathbf{u}\nabla\varphi + \sigma\varphi = f \quad \text{in } \Omega, \quad \varphi = 0 \quad \text{on } \Sigma_{\text{in}},$$

$$(1.28) \quad \mathcal{L}^*\varphi^* := -\operatorname{div}(\varphi^*\mathbf{u}) + \sigma\varphi^* = f \quad \text{in } \Omega, \quad \varphi^* = 0 \quad \text{on } \Sigma_{\text{out}}.$$

The bounded functions  $\varphi, \varphi^*$  are called the generalized solutions to problems (1.27), (1.28), respectively, if the integral identities

$$(1.29) \quad \int_{\Omega} (\varphi \mathcal{L}^* \zeta^* - f \zeta^*) dx = 0, \quad \int_{\Omega} (\varphi^* \mathcal{L} \zeta - f \zeta) dx = 0,$$

hold true for all test functions  $\zeta^*, \zeta \in C(\Omega) \cap H^{1,1}(\Omega)$ , respectively, such that  $\zeta^* = 0$  on  $\Sigma_{\text{out}}$  and  $\zeta = 0$  on  $\Sigma_{\text{in}}$ .

**Theorem 1.3.** Assume that  $\Sigma$  and  $\mathbf{U}$  satisfy conditions (H1)-(H3), the exponents  $s, r$  satisfy the inequalities

$$(1.30) \quad 1/2 < s \leq 1, \quad 1 < r < 3/(2s-1),$$

the vector field  $\mathbf{u}$  belongs to the class  $C^1(\Omega)$  and satisfies the boundary condition

$$(1.31) \quad \mathbf{u} = \mathbf{U} \text{ on } \Sigma, \quad \mathbf{u} = 0 \text{ on } \partial S.$$

Then there are positive constants  $\sigma^*$  and  $\delta^*$  depending only on  $\Sigma$ ,  $\mathbf{U}$ ,  $s, r$ , and  $\|\mathbf{u}\|_{C^1(\Omega)}$  such that:

(i) For any  $\sigma > \sigma^*$  and  $f \in H^{s,r}(\Omega) \cap L^\infty(\Omega)$ , problem (1.27) has the unique solution  $\varphi \in H^{s,r}(\Omega) \cap L^\infty(\Omega)$  satisfying the inequalities

$$(1.32) \quad \|\varphi\|_{H^{s,r}(\Omega)} \leq C\|f\|_{H^{s,r}(\Omega)}, \quad \|\varphi\|_{L^\infty(\Omega)} \leq \sigma^{-1}\|f\|_{L^\infty(\Omega)}.$$

(ii) If, in addition,  $\|\operatorname{div} \mathbf{u}\|_{H^{s,r}(\Omega)} + \|\operatorname{div} \mathbf{u}\|_{L^\infty(\Omega)} \leq \delta^*$ , problem (1.28) has a unique solution  $\varphi^* \in H^{s,r}(\Omega) \cap L^\infty(\Omega)$ , which admits the estimates

$$(1.33) \quad \|\varphi^*\|_{H^{s,r}(\Omega)} \leq C\|f\|_{H^{s,r}(\Omega)}, \quad \|\varphi^*\|_{L^\infty(\Omega)} \leq (\sigma - \delta^*)^{-1}\|f\|_{L^\infty(\Omega)}.$$

The constant  $C$  depends only on  $\|\mathbf{u}\|_{C^1(\Omega)}$ ,  $r, s, \sigma, \mathbf{U}$  and  $\Omega$ .

Since for  $sr > 3$ , the embeddings  $X^{s,r} \hookrightarrow C(\Omega)$ ,  $Y^{s,r} \hookrightarrow C^1(\Omega)$  are bounded, we have the following result on solvability of problems (1.27), (1.28) in space  $X^{s,r}$ .

**Corollary 1.4.** Assume that  $sr > 3$  and the vector field  $\mathbf{u}$  has the representation  $\mathbf{u} = \mathbf{u}_0 + \mathbf{v}$ , where  $\mathbf{u}_0 \in C^\infty(\Omega)^3$  is a solution to problem (1.11). Then there exist  $\tau^* \in (0, 1]$  and  $\sigma^*$ , depending only on  $\Sigma, \mathbf{u}_0$  and  $s, r$ , such that for all  $\mathbf{v}$  with  $\|\mathbf{v}\|_{Y^{s,r}} \leq \tau^*$ ,  $\sigma > \sigma^*$ , and  $f \in X^{s,r}$ , each of problems (1.27) and (1.28) has the unique solution satisfying the inequalities

$$(1.34) \quad \|\varphi\|_{X^{s,r}} \leq C\|f\|_{X^{s,r}}, \quad \|\varphi^*\|_{X^{s,r}} \leq C\|f\|_{X^{s,r}}.$$

*Existence and uniqueness theory.* The second main result of the paper concerns the existence and local uniqueness of solutions to problem (1.13). Denote by  $E$  the closed subspace of the Banach space  $Y^{s,r}(\Omega)^3 \times X^{s,r}(\Omega)^2$  in the following form

$$(1.35) \quad E = \{\vartheta = (\mathbf{v}, \pi, \varphi) : \mathbf{v} = 0 \text{ on } \partial\Omega, \quad \varphi = 0 \text{ on } \Sigma_{\text{in}}, \quad \Pi\pi = \pi\},$$

and denote by  $\mathcal{B}_\tau \subset E$  the closed ball of radius  $\tau$  centered at 0. Next, note that for  $sr > 3$ , elements of the ball  $\mathcal{B}_\tau$  satisfy the inequality

$$(1.36) \quad \|\mathbf{v}\|_{C^1(\Omega)} + \|\pi\|_{C(\Omega)} + \|\varphi\|_{C(\Omega)} \leq c_e(r, s, \Omega)\|\vartheta\|_E \leq c_e\tau,$$

where the norm in  $E$  is defined by

$$\|\vartheta\|_E = \|\mathbf{v}\|_{Y^{s,r}(\Omega)} + \|\pi\|_{X^{s,r}(\Omega)} + \|\varphi\|_{X^{s,r}(\Omega)}.$$

**Theorem 1.5.** Assume that the surface  $\Sigma$  and given vector field  $\mathbf{U}$  satisfy conditions (H1)-(H3). Furthermore, let  $\sigma^*, \tau^*$  be constants given by Corollary 1.4, and let positive numbers  $r, s, \sigma$  satisfy the inequalities

$$(1.37) \quad 1/2 < s \leq 1, \quad 1 < r < 3/(2s-1), \quad sr > 3, \sigma > \sigma^*.$$

Then there exists  $\tau_0 \in (0, \tau^*]$ , depending only on  $\mathbf{U}, \Omega, r, s, \sigma$ , such that for all

$$(1.38) \quad \tau \in (0, \tau_0], \quad \lambda^{-1}, R \in (0, \tau^2], \quad \|\mathbf{N} - \mathbf{I}\|_{C^2(\Omega)} \leq \tau^2,$$

problem (1.13), with  $\mathbf{u}_0$  given by (1.11), has a unique solution  $\vartheta \in \mathcal{B}_\tau$ . Moreover, the auxiliary function  $\zeta$  and the constants  $\varkappa, m$  admit the estimates

$$(1.39) \quad \|\zeta\|_{X^{s,r}} + |\varkappa| \leq c, \quad |m| \leq c\tau < 1,$$

where the constant  $c$  depends only on  $\mathbf{U}, \Omega, r, s$  and  $\sigma$ .

*Shape derivatives of solutions.* Theorem 1.5 guarantees the existence and uniqueness of solutions to problem (1.13) for all  $\mathbf{N}$  close to the identity matrix  $\mathbf{I}$ . The totality of such solutions can be regarded as the mapping from  $\mathbf{N}$  to the solution of the Navier-Stokes equations. The natural question is the smoothness properties of this mapping, in particular its differentiability. With application to shape optimization problems in mind, we consider the particular case where the matrices  $\mathbf{N}$  depend on the small parameter  $\varepsilon$  and have representation (1.6). We assume that  $C^1$  norms of the matrix-valued functions  $\mathbf{D}$  and  $\mathbf{D}_1(\varepsilon)$  in (1.6) have a majorant independent of  $\varepsilon$ . By virtue of Theorem 1.5, there are the positive constants  $\varepsilon_0$  and  $\tau$  such that for all sufficiently small  $R, \lambda^{-1}$  and  $\varepsilon \in [0, \varepsilon_0]$ , problem (1.13) with  $\mathbf{N} = \mathbf{N}(\varepsilon)$  has a unique solution  $\vartheta(\varepsilon) = (\mathbf{v}(\varepsilon), \pi(\varepsilon), \varphi(\varepsilon)), \zeta(\varepsilon), m(\varepsilon)$ , which admits the estimate

$$(1.40) \quad \|\vartheta(\varepsilon)\|_E + |m(\varepsilon)| \leq c\tau, \quad \|\zeta(\varepsilon)\|_{X^{s,r}} \leq c,$$

where the constant  $c$  is independent of  $\varepsilon$ , and the Banach space  $E$  is defined by (1.35). Denote the solution for  $\varepsilon = 0$  by  $(\vartheta(0), m(0), \zeta(0))$  by  $(\vartheta, m, \zeta)$ , and define the finite differences with respect to  $\varepsilon$

$$(\mathbf{w}_\varepsilon, \omega_\varepsilon, \psi_\varepsilon) = \varepsilon^{-1}(\vartheta - \vartheta(\varepsilon)), \quad \xi_\varepsilon = \varepsilon^{-1}(\zeta - \zeta(\varepsilon)), \quad n_\varepsilon = \varepsilon^{-1}(m - m(\varepsilon)).$$

Formal calculations shows that the limit  $(\mathbf{w}, \omega, \psi, \xi, n) = \lim_{\varepsilon \rightarrow 0} (\mathbf{w}_\varepsilon, \omega_\varepsilon, \psi_\varepsilon, \xi_\varepsilon, n_\varepsilon)$  is a solution to linearized equations

$$(1.41) \quad \begin{aligned} \Delta \mathbf{w} - \nabla \omega &= R \mathcal{C}_0(\mathbf{w}, \psi) + \mathcal{D}_0(\mathbf{D}) \text{ in } \Omega, \\ \operatorname{div} \mathbf{w} &= b_{21}^0 \psi - b_{22}^0 \omega + b_{23}^0 n + b_{30}^0 \mathfrak{d} \text{ in } \Omega, \\ \mathbf{u} \nabla \psi + \sigma \psi &= -\mathbf{w} \cdot \nabla \varphi + b_{11}^0 \psi + b_{12}^0 \omega + b_{13}^0 n + b_{10}^0 \mathfrak{d} \text{ in } \Omega, \\ -\operatorname{div}(\mathbf{u} \xi) + \sigma \xi &= \operatorname{div}(\zeta \mathbf{w}) + \sigma \mathfrak{d} \text{ in } \Omega, \\ \mathbf{w} &= 0 \text{ on } \partial \Omega, \quad \psi = 0 \text{ on } \Sigma_{\text{in}}, \quad \xi = 0 \text{ on } \Sigma_{\text{out}}, \\ \omega - \Pi \omega &= 0, \quad n = \varkappa \int_{\Omega} (b_{31}^0 \psi + b_{32}^0 \omega + b_{34}^0 \xi + b_{30}^0 \mathfrak{d}) dx, \end{aligned}$$

where  $\mathfrak{d} = 1/2 \operatorname{Tr} \mathbf{D}$ , the variable coefficients  $b_{ij}^0$  and the operators  $\mathcal{C}_0, \mathcal{D}_0$ , are defined by the formulae

$$(1.42) \quad \begin{aligned} b_{11}^0 &= \Psi[\vartheta] - \varrho H'(\varphi) + m - \frac{2\sigma}{\varrho_0} \varphi, \quad b_{12}^0 = \lambda^{-1} \varrho, \quad b_{13}^0 = \varrho, \\ b_{10}^0 &= \varrho \Psi[\vartheta] - \frac{\sigma}{\varrho_0} \varphi^2 - \sigma \varphi + m \varrho, \quad b_{21}^0 = \frac{\sigma}{\varrho_0} \psi_0 + H'(\varphi), \\ b_{22}^0 &= -\lambda^{-1}, \quad b_{23}^0 = -1, \quad b_{20}^0 = \sigma \varphi \varrho_0^{-1} - \Psi[\vartheta] - m, \\ b_{31}^0 &= \varrho_0^{-1} \zeta \left( \Psi[\vartheta] - \varrho H'(\varphi) - \frac{2\sigma}{\varrho_0} \varphi \right) - H'(\varphi) + m \varrho_0^{-1} \zeta, \\ b_{32}^0 &= (\lambda \varrho_0)^{-1} \varrho \zeta b_{12}^0 + \lambda^{-1}, \quad b_{34}^0 = \varrho_0^{-1} \Psi_1[\vartheta] + m(1 + \varrho_0^{-1} \varphi) \\ b_{30}^0 &= \varrho_0^{-1} \zeta (\mathfrak{d}_0 - m \varrho) + \Psi[\vartheta] - m(1 - \zeta - \varrho_0^{-1} \zeta \varphi), \end{aligned}$$

$$(1.43) \quad \mathcal{C}_0(\psi, \mathbf{w}) = R\psi \mathbf{u} \nabla \mathbf{u} + R\rho \mathbf{w} \nabla \mathbf{u} + R\rho \mathbf{u} \nabla \mathbf{w},$$

$$(1.44) \quad \mathcal{D}_0(\mathbf{D}) = R\mathbf{u} \nabla (\mathbf{D}\mathbf{u}) + R\mathbf{D}^*(\mathbf{u} \nabla \mathbf{u}) + \operatorname{div} \left( (\mathbf{D} + \mathbf{D}^*) \nabla \mathbf{u} - \frac{1}{2} \operatorname{Tr} \mathbf{D} \nabla \mathbf{u} \right) - \mathbf{D} \Delta \mathbf{u} - \Delta (\mathbf{D}\mathbf{u}).$$

The justification of the formal procedure meets the serious problems, since the smoothness of solutions to problem (1.13) is not sufficient for the well-posedness of problem (1.41) in the standard weak formulation. In order to cope with this difficulty we define *very weak solutions* to problem (1.41). The construction of such solutions is based on the following lemma, the proof is given in Appendix. The lemma is given in  $\mathbb{R}^d$ , for our application  $d = 3$ .

**Lemma 1.6.** *Let  $\Omega \subset \mathbb{R}^d$  be a bounded domain with the Lipschitz boundary, let exponents  $s$  and  $r$  satisfy the inequalities  $sr > d$ ,  $1/2 \leq s \leq 1$  and  $\varphi, \varsigma \in H^{s,r}(\Omega) \cap H^{1,2}(\Omega)$ ,  $\mathbf{w} \in \mathcal{H}_0^{1-s,r'}(\Omega) \cap H_0^{1,2}(\Omega)$ . Then there is a constant  $c$  depending only on  $s, r$  and  $\Omega$ , such that the trilinear form*

$$\mathfrak{B}(\mathbf{w}, \varphi, \varsigma) = - \int_{\Omega} \varsigma \mathbf{w} \cdot \nabla \varphi \, dx$$

*satisfies the inequality*

$$(1.45) \quad |\mathfrak{B}(\mathbf{w}, \varphi, \varsigma)| \leq c \|\mathbf{w}\|_{\mathcal{H}_0^{1-s,r'}(\Omega)} \|\varphi\|_{H^{s,r}(\Omega)} \|\varsigma\|_{H^{s,r}(\Omega)},$$

*and can be continuously extended to  $\mathfrak{B} : \mathcal{H}_0^{1-s,r'}(\Omega)^d \times H^{s,r}(\Omega)^2 \mapsto \mathbb{R}$ . In particular, we have  $\varsigma \nabla \varphi \in \mathcal{H}^{s-1,r}(\Omega)$  and  $\|\varsigma \nabla \varphi\|_{H^{s-1,r}(\Omega)} \leq c \|\varphi\|_{H^{s,r}(\Omega)} \|\varsigma\|_{H^{s,r}(\Omega)}$ .*

**Definition 1.7.** *The vector field  $\mathbf{w} \in \mathcal{H}_0^{1-s,r'}(\Omega)^3$ , functionals  $(\omega, \psi, \xi) \in \mathbb{H}^{-s,r'}(\Omega)^3$  and constant  $n$  are said to be a weak solution to problem (1.41), if  $\langle \omega, 1 \rangle = 0$  and the identities*

$$(1.46) \quad \begin{aligned} & \int_{\Omega} \mathbf{w} \left( \mathbf{H} - R\rho \nabla \mathbf{u} \cdot \mathbf{h} + R\rho \nabla \mathbf{h}^* \mathbf{u} \right) dx - \mathfrak{B}(\mathbf{w}, \varphi, \varsigma) - \mathfrak{B}(\mathbf{w}, v, \zeta) + \\ & \langle \omega, G - b_{12}^0 \varsigma - b_{22}^0 g - \kappa b_{32}^0 \rangle + \langle \psi, F - b_{11}^0 \varsigma - b_{21}^0 g - \kappa b_{31}^0 - R\mathbf{u} \cdot \nabla \mathbf{u} \cdot \mathbf{h} \rangle + \\ & \langle \xi, M - \kappa b_{34}^0 \rangle + n(1 - \langle 1, b_{13}^0 \varsigma \rangle) = \\ & \langle \mathfrak{d}, b_{10}^0 \varsigma + b_{20}^0 g + \kappa b_{30}^0 + \sigma v \rangle + \langle \mathcal{D}_0, \mathbf{h} \rangle. \end{aligned}$$

*hold true for all  $(\mathbb{H}, G, F, M) \in (C^\infty(\Omega))^6$  such that  $G = \Pi G$ . Here  $\mathfrak{d} = 1/2 \operatorname{Tr} \mathbf{D}$ , the test functions  $\mathbf{h}, g, \varsigma, v$  are defined by the solutions to adjoint problems*

$$(1.47) \quad \Delta \mathbf{h} - \nabla g = \mathbb{H}, \quad \operatorname{div} \mathbf{h} = G, \quad \mathcal{L}^* \varsigma = F, \quad \mathcal{L} v = M \text{ in } \Omega,$$

$$(1.48) \quad \mathbf{h} = 0 \text{ on } \partial\Omega, \quad \Pi g = g, \varsigma = 0 \text{ on } \Sigma_{\text{out}}, \quad v = 0 \text{ on } \Sigma_{\text{in}}.$$

We are now in a position to formulate the third main result of this paper.

**Theorem 1.8.** *Under the above assumptions,*

$$(1.49) \quad \begin{aligned} & \mathbf{w}_\varepsilon \rightarrow \mathbf{w} \text{ weakly in } \mathcal{H}_0^{1-s,r'}(\Omega), \quad n_\varepsilon \rightarrow n \text{ in } \mathbb{R}, \\ & \psi_\varepsilon \rightarrow \psi, \quad \omega_\varepsilon \rightarrow \omega, \quad \xi_\varepsilon \rightarrow \xi \text{ (*)-weakly in } \mathbb{H}^{-s,r'}(\Omega) \text{ as } \varepsilon \rightarrow 0, \end{aligned}$$

*where the limits, vector field  $\mathbf{w}$ , functionals  $\psi, \omega, \xi$ , and the constant  $n$  are given by the weak solution to problem (1.41).*

Note that the matrices  $\mathbf{N}(\varepsilon)$  defined by equalities (1.5) meet all requirements of Theorem 1.8, and in the special case we have in representation (1.6)

$$(1.50) \quad \mathbf{D}(x) = \operatorname{div} \mathbf{T}(x) \mathbf{I} - \mathbf{T}'(x).$$

Therefore, Theorem 1.8, together with the formulae (1.3) and (1.10), imply the existence of the shape derivative for the drag functional at  $\varepsilon = 0$ . Straightforward calculations lead to the following result.

**Theorem 1.9.** *Under the assumptions of Theorem 1.8, there exists the shape derivative*

$$\frac{d}{d\varepsilon} J_D(S_\varepsilon) \Big|_{\varepsilon=0} = L_e(\mathbf{T}) + L_u(\mathbf{w}, \omega, \psi),$$

where the linear forms  $L_e$  and  $L_u$  are defined by the equalities

$$\begin{aligned} L_e(\mathbf{T}) &= \int_{\Omega} \operatorname{div} \mathbf{T} (\nabla \mathbf{u} + \nabla \mathbf{u}^* - \operatorname{div} \mathbf{u} \mathbf{I}) \mathbf{U}_{\infty} dx - \\ &\int_{\Omega} [\nabla \mathbf{u} + \nabla \mathbf{u}^* - \operatorname{div} \mathbf{u} - q \mathbf{I} - R \varrho \mathbf{u} \otimes \mathbf{u}] \mathbf{D} \nabla \eta \cdot \mathbf{U}_{\infty} dx - \\ &\int_{\Omega} [\mathbf{D}^* \nabla \mathbf{u} + \nabla \mathbf{u}^* \mathbf{D} + \nabla (\mathbf{D} \mathbf{u}) + \nabla (\mathbf{D} \mathbf{u})^*] \nabla \eta \cdot \mathbf{U}_{\infty} dx \end{aligned}$$

and

$$\begin{aligned} L_u(\mathbf{w}, \omega, \psi) &= \int_{\Omega} \mathbf{w} [\Delta \eta \mathbf{U}_{\infty} + R \varrho (\mathbf{u} \cdot \nabla \eta) \mathbf{U}_{\infty} + R \varrho (\mathbf{u} \cdot \mathbf{U}_{\infty}) \nabla \eta] dx + \\ &\langle \omega, \nabla \eta \cdot \mathbf{U}_{\infty} \rangle + R \langle \psi, (\mathbf{u} \cdot \nabla \eta) (\mathbf{u} \cdot \mathbf{U}_{\infty}) \rangle. \end{aligned}$$

While  $L_e$  depends directly on the vector field  $\mathbf{T}$ , the linear form  $L_u$  depends on the weak solution  $(\mathbf{w}, \psi, \omega)$  to problem (1.41), thus depends on the *direction*  $\mathbf{T}$  in a very implicit manner, which is inconvenient for applications. In order to cope with this difficulty, we define the *adjoint state*  $\mathbf{Y} = (\mathbf{h}, g, \varsigma, v, l)^{\top}$  given as a solution to the linear equation

$$(1.51) \quad \mathfrak{L} \mathbf{Y} - \mathfrak{U} \mathbf{Y} - \mathfrak{V} \mathbf{Y} = \Theta,$$

supplemented with boundary conditions (1.48). Here the operators  $\mathfrak{L}$ ,  $\mathfrak{U}$ ,  $\mathfrak{V}$  and the vector field  $\Theta$  are defined by

$$\begin{aligned} \mathfrak{L} &= \begin{pmatrix} \Delta & -\nabla & 0 & 0 & 0 \\ \operatorname{div} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{L}^* & 0 & 0 \\ 0 & 0 & 0 & \mathcal{L} & 0 \\ 0 & 0 & -\mathbb{B}_{13} & 0 & 1 \end{pmatrix}, \quad \mathfrak{U} = \begin{pmatrix} 0 & 0 & -\nabla \varphi & -\zeta \nabla & 0 \\ 0 & 0 & \Pi_{21} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ \mathfrak{V} &= \begin{pmatrix} R \varrho (\nabla \mathbf{u} - \mathbf{u} \nabla) & 0 & 0 & 0 & 0 \\ 0 & -\lambda^{-1} \Pi & 0 & 0 & \varkappa \Pi b_{32}^0 \\ R \mathbf{u} \cdot \nabla \mathbf{u} & b_{12}^0 & b_{11}^0 & 0 & \varkappa b_{31}^0 \\ 0 & 0 & 0 & 0 & \varkappa b_{34}^0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \Theta &= (\Delta \eta \mathbf{U}_{\infty} + R \varrho (\nabla \eta \otimes \mathbf{U}_{\infty} + \mathbf{U}_{\infty} \otimes \nabla \eta) \mathbf{u}, \Pi (\nabla \eta \cdot \mathbf{U}_{\infty}), R (\mathbf{u} \nabla \eta) (\mathbf{u} \mathbf{U}_{\infty}), 0, 0), \\ \Pi_{2i}(\cdot) &= \Pi(b_{2i}^0(\cdot)), \quad \mathbb{B}_{13}(\cdot) = \langle 1, b_{13}^0(\cdot) \rangle \end{aligned}$$

The following theorem guarantees the existence of the adjoint state and gives the expression of the shape derivative for the drag functional in terms of the vector field  $\mathbf{T}$ .

**Theorem 1.10.** *Let a given solution  $\vartheta \in \mathcal{B}_\tau$ ,  $(\zeta, m) \in X^{s,r} \times \mathbb{R}$ , to problem (1.13) meets all requirements of Theorem 1.5. Then there exists positive constant  $\tau_1$  (depending only on  $\mathbf{U}$ ,  $\Omega$  and  $r, s$ ) such that, if  $\tau \in (0, \tau_1]$  and  $R\lambda^{-1} \leq \tau_1^2$ , then there exists a unique solution  $\mathbf{Y} \in (Y^{s,r})^3 \times (X^{s,r})^3 \times \mathbb{R}$  to problem (1.51), (1.48). The form  $L_u$  has the representation*

$$(1.52) \quad L_u(\mathbf{w}, \psi, \omega) = \int_{\Omega} [\operatorname{div} \mathbf{T}(b_{10}^0 \varsigma + b_{20}^0 g + \sigma v + \kappa b_{30}^0 l) + \mathcal{D}_0(\operatorname{div} \mathbf{T} - \mathbf{T}') \mathbf{h}] dx$$

where the coefficients  $b_{ij}^0$  and the operator  $\mathcal{D}_0$  are defined by the formulae (1.42), (1.44).

*Method and structure of the paper.* The following aspects of our method deserve brief description.

- Extended form (1.13), of the governing equations which allows to cope with the mass control problem.
- The splitting of the boundary value problem for the transport equation into two parts: the local problem in the vicinity of inlet, and the global problem with the modified vector field  $\tilde{\mathbf{u}}$  and the empty inlet  $\tilde{\Sigma}_{in}$ .
- The estimates of solutions to the model problem (4.24) in the fractional Sobolev spaces, which can not be obtained by the interpolation method.
- The very weak formulation of linearized equations introduced to assure the existence of shape derivatives.

Now, we can explain the organization of the paper. Section 2 is devoted to the proof of Theorem 1.5. First of all, we establish the existence of solution to problem (1.13) using Schauder fixed point theorem. Next, we consider the linear equations for difference of two solutions  $(\mathbf{v}_i, \varphi_i)$ ,  $i = 1, 2$ , corresponding to arbitrary matrix-valued functions  $\mathbf{N}_i$ . Using Theorem 1.3 we deduce the weak formulation of boundary value problem for linearized equations. The main result of this section is Theorem 2.3 which show that solutions of the linearized problem are stable with respect perturbations of data in the dual Sobolev space. This result implies the local uniqueness of solutions to problem (1.13). In section 3 we exploit Theorem 2.3 to prove the existence of the shape derivative of solutions. The last section is devoted to the proof of Theorem 1.3.

## 2. EXISTENCE AND UNIQUENESS OF LOCAL SOLUTIONS. PROOF OF THEOREM 1.5

**2.1. Existence theory.** In this paragraph we establish the local solvability of problem (1.13) and prove the first part of Theorem 1.5. In our notation,  $c$  denotes generic constants, which are different in different places and depend only on  $\Omega$ ,  $\mathbf{U}$ ,  $\sigma$  and  $r, s$ . The proof is based on the following lemma which furnishes the regularity properties of composed functions. Let us consider functions  $u, v : \Omega \mapsto B_K$ , where  $B_K = \{\mathbf{x} : |\mathbf{x}| \leq K\} \subset \mathbb{R}^3$  is the ball of radius  $K$  centered at 0.



**Lemma 2.1.** *Assume that  $u, v \in X^{s,r}$ ,  $s \in (0, 1]$ ,  $sr > 3$ , and  $f \in C^3(\Omega \times B_K)$ . Then we have*

$$(2.1) \quad \|f(\cdot, u)\|_{X^{s,r}} \leq c(r, s) \|f\|_{C^1(\Omega \times B_K(0))} (1 + \|u\|_{X^{s,r}}),$$

$$(2.2) \quad \|f(\cdot, v) - f(\cdot, u)\|_{X^{s,r}} \leq c(r, s) \|f\|_{C^2(\Omega \times B_K)} (1 + \|u\|_{X^{s,r}} + \|v\|_{X^{s,r}}) \|u - v\|_{X^{s,r}}$$

*Proof.* In order to prove (2.1) it suffices to note that

$$|f(x, u(x)) - f(y, u(y))|^r \leq c(r) \|f\|_{C^1(\Omega \times B_K)}^r (|x - y|^r + |u(x) - u(y)|^r),$$

which, in view of the inequality,

$$\int_{\Omega \times \Omega} |x - y|^{r-3-rs} dx dy \leq c(r, s),$$

yields

$$|f(\cdot, u)|_{s,r,\Omega} \leq c(r, s) \|f\|_{C^1(\Omega \times B_K(0))} (1 + |u|_{s,r,\Omega}).$$

On the other hand, we have

$$\|\nabla f(\cdot, u)\|_{L^2(\Omega)} \leq \|f\|_{C^1(\Omega \times B_K(0))} \|\nabla u\|_{L^2(\Omega)}.$$

Combining obtained inequalities we get (2.1). It remains to note that (2.2) follows from (2.1) and the Hadamard formula for the first order expansion of  $f$ .  $\square$

Fix sufficiently small positive  $\tau$ , such that

$$(2.3) \quad c_e \tau < \delta^*,$$

where  $\delta^*$  is the constant determined in Corollary 1.4, and  $c_e$  is the constant from inequality (1.36). By virtue of Corollary 1.4, there is  $\sigma^*$ , depending only on  $\Omega$ ,  $\mathbf{U}$ , and  $r, s$ , such that for all  $\vartheta \in \mathcal{B}_\tau$  and  $\sigma > \sigma^*$ , problems (1.27) and (1.28) have solutions satisfying inequalities (1.34). Finally fix an arbitrary  $\sigma > \sigma^*$ .

We solve problem (1.13) by an application of the Schauder fixed point Theorem in the following framework. Choose an arbitrary element  $\vartheta \in \mathcal{B}_\tau$ . As it is mentioned above, the problem

$$(2.4) \quad \mathbf{u} \cdot \nabla \varphi_1 + \sigma \varphi_1 = \Psi_1[\vartheta] + m \mathbf{g} \varrho \text{ in } \Omega, \quad \varphi_1 = 0 \text{ on } \Sigma_{\text{in}}$$

has a unique solution satisfying the inequality

$$(2.5) \quad \|\varphi_1\|_{X^{s,r}} \leq c(\Omega, U, \sigma, r, s) (\|\Psi_1[\vartheta]\|_{X^{s,r}} + |m|).$$

Next, define  $\mathbf{v}_1$  and  $\pi_1$  to be the solutions of the boundary problem for the Stokes equations

$$(2.6) \quad \begin{aligned} \Delta \mathbf{v}_1 - \nabla \pi_1 &= \mathcal{A}(\mathbf{u}) + R\mathcal{B}(\varrho, \mathbf{u}, \mathbf{u}) \equiv F[\vartheta] \text{ in } \Omega \\ \varrho_0 \operatorname{div} \mathbf{v}_1 &= \Pi(\mathbf{g} \sigma \varphi_1 - \mathbf{g} \varrho_0 \Psi[\vartheta] - \mathbf{g} m \varrho_0) \text{ in } \Omega, \\ \mathbf{v}_1 &= 0 \text{ on } \partial\Omega, \quad \pi_1 - \Pi \pi_1 = 0, \end{aligned}$$

where  $m$  is given by (1.13c). By Lemma 1.2, this problem admits a unique solution such that

$$(2.7) \quad \|\mathbf{v}_1\|_{Y^{s,r}} + \|\pi_1\|_{X^{s,r}} \leq c(\|F[\vartheta]\|_{Z^{s,r}} + \|\Psi[\vartheta]\|_{X^{s,r}} + \|\varphi_1\|_{X^{s,r}} + |m|).$$

Equations (2.4), (2.6), (1.13c), define the mapping  $\Xi : \vartheta \rightarrow \vartheta_1 = (\mathbf{v}_1, \pi_1, \varphi_1)$ . We claim that for a suitable choice of the constant  $\tau$ ,  $\Xi$  is a weakly continuous auto-morphism of the ball  $\mathcal{B}_\tau$ . We begin with the estimates for nonlinear operators present in (2.4). Fix an arbitrary  $\vartheta \in \mathcal{B}_\tau$ . Applying inequality (2.2) from Lemma

2.1 to the function  $H$  which is a part of  $\Psi[\vartheta]$ , we obtain  $\|H(\varphi)\|_{X^{s,r}} \leq c\tau^2$ , which leads to the estimate

$$(2.8) \quad \|\Psi[\vartheta]\|_{X^{s,r}} \leq \frac{c}{\lambda}(\|q_0\|_{C^1(\Omega)} + \|\pi\|_{X^{s,r}}) + c\tau^2 \leq c/\lambda + c\tau^2 \leq c\tau^2.$$

Since, under assumptions of Theorem 1.5,  $X^{s,r}(\Omega)$  is a Banach algebra and  $\|\varrho\|_{X^{s,r}} \leq c + \|\varphi\|_{X^{s,r}} \leq \text{const}$ , we conclude from this and (2.5) that

$$(2.9) \quad \|\varphi_1\|_{X^{s,r}} \leq c/\lambda + c\tau^2 + c|m| \leq c\tau^2 + c|m|.$$

In order to estimate the right hand side of the first equation in (2.6) we introduce the vector function  $\mathbf{z} = (\mathbf{v}, \nabla \mathbf{v}, \pi, \varphi)$  and proceed as follows. It can be easily seen that  $\|\mathbf{z}\|_{X^{s,r}} \leq \|\vartheta\|_E \leq \tau$ , and  $|\mathbf{z}| \leq c\tau$ . Recall that the operator  $\mathcal{B}$  constitutes a cubic polynomial of  $u$  and  $\varrho$ . By Lemma 2.1, we have

$$(2.10) \quad \|\mathcal{B}(\varrho, \mathbf{u}, \mathbf{u})\|_{X^{s,r}} \leq cR(1 + \|\varrho\|_{X^{s,r}} + \|\mathbf{z}\|_{X^{s,r}}) \leq c\tau^2(1 + \tau) \text{ in } \mathcal{B}_\tau.$$

Next, note that

$$\|\mathcal{A}(\mathbf{u})\|_{Z^{s,r}} \leq c(\|\mathbf{g} - \mathbf{1}_\Omega\|_{C^2(\Omega)} + \|\mathbf{N} - \mathbf{I}\|_{C^2(\Omega)})(1 + \|\nabla \mathbf{u}\|_{Y^{s,r}}) \leq c\tau^2\|\mathbf{u}\|_{Y^{s,r}},$$

which along with (1.38) and (2.10) implies

$$(2.11) \quad \|F[\vartheta]\|_{Z^{s,r}} \leq c\tau^2(1 + \tau) \text{ in } \mathcal{B}_\tau.$$

Combining inequalities (2.8) and (2.9) we get the estimate

$$\|\sigma\varphi_1 + \Psi[\vartheta]\|_{X^{s,r}} \leq c\tau^2.$$

From this, (2.11), (2.7) and Lemma 1.2 we finally obtain

$$(2.12) \quad \|\mathbf{v}_1\|_{Y^{s,r}} + |\pi_1|_{X^{s,r}} \leq c\tau^2 + c|m|.$$

It remains to estimate  $m$ . Recall that the vector field  $\mathbf{u}$  and parameter  $\sigma$  meet all requirements of Corollary 1.4. Therefore, problem (1.13b) has the unique solution  $\zeta \in H^{s,r}(\Omega)$  for all  $s, r$  satisfying (1.37). In particular, inequalities (1.34) yield the estimate  $\|\zeta\|_{X^{s,r}} \leq c$ . Since, by virtue of (1.37), the pair  $s = 2/3$ ,  $r = 6$  is admissible and the embedding  $H^{2/3,6}(\Omega) \hookrightarrow C^{1/6}(\Omega)$  is bounded, estimates (1.32) and (1.33) for  $rs > 3$  yield

$$(2.13) \quad \|\zeta\|_{C^{1/6}(\Omega)} + \|\zeta\|_{H^{1,2}(\Omega)} \leq C(\mathbf{U}, \Omega, \sigma).$$

Recalling that  $\text{div } \mathbf{u} = \text{div } \mathbf{v}$ , we obtain  $|\text{div } \mathbf{u}| \leq c_e\tau$ . From this, inequality  $|\mathbf{g}| \leq 1 + c\tau^2$ , and maximum principle (1.33) we conclude that

$$(2.14) \quad \|\zeta\|_{C(\Omega)} \leq (1 + c\tau^2)(1 - \sigma^{-1}c\tau)^{-1} \leq (1 - c\tau)^{-1},$$

which leads to the following estimate

$$|1 - \zeta| \leq c\tau(1 - c\tau)^{-1}.$$

Now we can estimate the right hand side of (1.13c). Rewrite the first integral in the form

$$\int_{\Omega} \mathbf{g}(1 - \zeta - \varrho_0^{-1}\zeta\varphi) dx = \int_{\Omega} (1 - \zeta)^+ dx + \int_{\Omega} (\mathbf{g} - 1)(1 - \zeta - \varrho_0^{-1}\zeta\varphi) dx - \int_{\Omega} ((1 - \zeta)^- + \varrho_0^{-1}\zeta\varphi) dx.$$

We have

$$|(\mathbf{g} - 1)(1 - \zeta - \varrho_0^{-1}\zeta\varphi)| \leq c\tau^2, \quad |(1 - \zeta)^- + \varrho_0^{-1}\zeta\varphi| \leq c_e\tau + c\tau(1 - c\tau)^{-1}.$$

On the other hand, we have  $\|(1 - \zeta)^+\|_{C^{1/6}(\Omega)} \leq c(\mathbf{U}, \Omega, \sigma)$  and  $(1 - \zeta)^+ = 1$  on  $\Sigma_{\text{out}}$ . Hence

$$\int_{\Omega} (1 - \zeta)^+ dx > \kappa(\mathbf{U}, \Omega, \sigma) > 0.$$

Thus, we get

$$\varkappa^{-1} \geq \kappa(1 - c\kappa^{-1}\tau(1 - c\tau)^{-1}).$$

In particular, there is positive  $\tau_0$  depending only on  $\mathbf{U}$ ,  $\Omega$  and  $\sigma$ , such that

$$|\varkappa| \leq c \text{ for all } \tau \leq \tau_0.$$

Repeating these arguments and using inequalities (2.8), (1.13c), we arrive at  $|m| \leq c\tau^2$ . Combining this estimate with (2.9) and (2.12), we finally obtain  $\|\vartheta_1\|_{X^{s,r}} \leq c\tau^2$ . Choose sufficiently small  $\tau_0 = \tau_0(\mathbf{U}, \Omega, \sigma)$ , such that  $c\tau_0^2 < \tau_0$ . Thus, for all  $\tau \leq \tau_0$ ,  $\Xi$  maps the ball  $\mathcal{B}_\tau$  into itself. Let us show that  $\Xi$  is weakly continuous. Choose an arbitrary sequence  $\vartheta_n \in \mathcal{B}_\tau$  such that  $\vartheta_n = (\mathbf{v}_n, \pi_n, \varphi_n)$  converges weakly in  $E$  to some  $\vartheta$ . Since the ball  $\mathcal{B}_\tau$  is closed and convex,  $\vartheta$  belongs to  $\mathcal{B}_\tau$ . Let us consider the corresponding sequences of the elements  $\vartheta_{1,n} = \Xi(\vartheta_n) \in \mathcal{B}_\tau$  and functions  $\zeta_n$ . There are subsequences  $\{\vartheta_{1,j}\} \subset \{\vartheta_{1,n}\}$  and  $\{\zeta_j\} \subset \{\zeta_n\}$  such that  $\vartheta_{1,j}$  converges weakly in  $E$  to some element  $\vartheta_1 \in \mathcal{B}_\tau$  and  $\zeta_j$  converges weakly in  $X^{s,r}$  to some function  $\zeta \in X^{s,r}$ . Since the embedding  $E \hookrightarrow C(\Omega)^5$  is compact, we have  $\vartheta_n \rightarrow \vartheta$ ,  $\vartheta_{1,j} \rightarrow \vartheta_1$  in  $C(\Omega)^5$ , and

$$\nabla \zeta_j \rightharpoonup \nabla \zeta \text{ weakly in } L^2(\Omega), \quad \zeta_j \rightarrow \zeta \text{ in } C(\Omega).$$

Substituting  $\vartheta_j$  and  $\vartheta_{1,j}$  into equations (2.4), (2.6), (1.13c) and letting  $j \rightarrow \infty$  we obtain that the limits  $\vartheta$  and  $\vartheta_1$  also satisfy (2.4), (2.6), (1.13c). Thus, we get  $\vartheta_1 = \Xi(\vartheta)$ . Since for given  $\vartheta$ , a solution to equations (2.4), (2.6) is unique, we conclude from this that all weakly convergent subsequences of  $\vartheta_{1,n}$  have the unique limit  $\vartheta_1$ . Therefore, the whole sequence  $\vartheta_{1,n} = \Xi(\vartheta_n)$  converges weakly to  $\Xi(\vartheta)$ . Hence the mapping  $\Xi : \mathcal{B}_\tau \mapsto \mathcal{B}_\tau$  is weakly continuous and, by virtue of the Schauder fixed-point theory, there is  $\vartheta \in \mathcal{B}(\tau)$  such that  $\vartheta = \Xi(\vartheta)$ .

It remains to prove that  $\vartheta$  is given by a solution to problem (1.13a). For  $\vartheta_1 = \vartheta$ , the only difference between problems (1.13a) and (2.6), (1.13c) is the presence of the projector  $\Pi$  in the right hand side of (2.6). Hence, it suffices to show that

$$(2.15) \quad \Pi(\varrho_0^{-1} \mathbf{g} \sigma \varphi - \mathbf{g} \Psi[\vartheta] - \mathbf{g} m) = \varrho_0^{-1} \mathbf{g} \sigma \varphi - \mathbf{g} \Psi[\vartheta] - \mathbf{g} m.$$

To this end we note that  $\varphi$  is a generalized solution to the transport equation

$$\mathbf{u} \cdot \nabla \varphi + \sigma \varphi = \Psi_1[\vartheta] + m \mathbf{g} \varrho.$$

Using  $\zeta$  as a test function and recalling the integral identity (1.29) we obtain

$$\sigma \int_{\Omega} \varphi \mathbf{g} dx = \int_{\Omega} \zeta (\Psi_1[\vartheta] + m \mathbf{g} \varrho) dx.$$

On the other hand, equality (1.13c) reads

$$\int_{\Omega} \zeta (\varrho_0^{-1} \Psi_1[\vartheta] + m \mathbf{g} (1 + \varphi \varrho_0^{-1})) dx = \int_{\Omega} (a \Psi[\vartheta] + \mathbf{g} m) dx.$$

Combining these equalities and noting that  $1 + \varrho_0^{-1} \varphi = \varrho / \varrho_0$  we obtain

$$\int_{\Omega} \left( \frac{\mathbf{g} \sigma}{\varrho_0} \varphi - \mathbf{g} \Psi[\vartheta] - m \mathbf{g} \right) dx = 0$$

which yields (2.15) and the proof of Theorem 1.5 is completed.  $\square$

**2.2. Uniqueness and stability.** In this paragraph we prove that, under the assumptions of Theorem 1.5, a solution to problem (1.13) is unique, and investigate in details the dependence of the solution on the matrix function  $\mathbf{N}$ .

*Weak formulation of linearized equations.* Assume that matrices  $\mathbf{N}_i$ ,  $i = 1, 2$ , satisfy conditions of Theorem 1.5, and denote by  $(\vartheta_i, \zeta_i, m_i) \in E \times X^{s,r} \times \mathbb{R}$ ,  $i = 1, 2$ , the corresponding solutions to problem (1.13). Recall that the solutions  $(\vartheta_i, \zeta_i, m_i)$ , together with the constants  $\kappa_i$ , satisfy the inequalities

$$(2.16) \quad |m_i| + \|\vartheta_i\|_E \leq c\tau, \quad |\kappa_i| + \|\zeta_i\|_{X^{s,r}} \leq c,$$

where the constant  $c$  depends only on  $\mathbf{U}, \Omega, r, s$ , and  $\sigma$ . We denote  $\mathbf{u}_i = \mathbf{u}_0 + \mathbf{v}_i$ ,  $i = 1, 2$ , the solutions to (1.9) for  $\mathbf{N}_i$ ,  $i = 1, 2$ . Now set

$$\mathfrak{d} = \mathfrak{g}_1 - \mathfrak{g}_2 \equiv \sqrt{\det \mathbf{N}_1} - \sqrt{\det \mathbf{N}_2},$$

$$\mathbf{w} = \mathbf{v}_1 - \mathbf{v}_2, \quad \omega = \pi_1 - \pi_2, \quad \psi = \varphi_1 - \varphi_2, \quad \xi = \zeta_1 - \zeta_2, \quad n = m_1 - m_2.$$

It follows from (1.13) that

$$(2.17) \quad \begin{aligned} \mathbf{u}_1 \nabla \psi + \sigma \psi &= -\mathbf{w} \cdot \nabla \varphi_2 + b_{11} \psi + b_{12} \omega + b_{13} n + b_{10} \mathfrak{d} \text{ in } \Omega, \\ \Delta \mathbf{w} - \nabla \omega &= \mathcal{A}_1(\mathbf{w}) + R\mathcal{C}_1(\psi, \mathbf{w}) + \mathcal{D} \text{ in } \Omega, \\ \operatorname{div} \mathbf{w} &= b_{21} \psi + b_{22} \omega + b_{23} n + b_{20} \mathfrak{d} \text{ in } \Omega, \\ -\operatorname{div}(\mathbf{u}_1 \xi) + \sigma \xi &= \operatorname{div}(\zeta_2 \mathbf{w}) + \sigma \mathfrak{d} \text{ in } \Omega, \\ \mathbf{w} &= 0 \text{ on } \partial\Omega, \quad \psi = 0 \text{ on } \Sigma_{\text{in}}, \quad \xi = 0 \text{ on } \Sigma_{\text{out}}, \\ \omega - \Pi \omega &= 0, \quad n = \kappa \int_{\Omega} (b_{31} \psi + b_{32} \omega + b_{34} \xi + b_{30} \mathfrak{d}) dx. \end{aligned}$$

Here the coefficients are given by the formulae

$$(2.18) \quad \begin{aligned} b_{11} &= \sigma(1 - \mathfrak{g}_2) + \mathfrak{g}_2 \Psi[\vartheta_1] - \mathfrak{g}_2 \varrho_2 \Phi_1(\varphi_1, \varphi_2) + \mathfrak{g}_2 m_2 - \frac{\sigma \mathfrak{g}_2}{\varrho_0} (\varphi_1 + \varphi_2), \\ b_{12} &= \lambda^{-1} \varrho_2 \mathfrak{g}_2, \quad b_{13} = \mathfrak{g}_2 \varrho_1, \quad b_{10} = \varrho_1 \Psi[\vartheta_1] - \frac{\sigma}{\varrho_0} \varphi_1^2 - \sigma \varphi_1 + m_1 \varrho_1, \\ b_{21} &= \mathfrak{g}_2 \left( \frac{\sigma}{\varrho_0} + \Phi_1(\varphi_1, \varphi_2) \right), \quad b_{22} = -\mathfrak{g}_2 / \lambda, \\ b_{23} &= -\mathfrak{g}_2, \quad b_{20} = \sigma \varphi_1 \varrho_0^{-1} - \Psi[\vartheta_1] - m_2, \\ b_{31} &= \varrho_0^{-1} \zeta_1 \left( \sigma(1 - \mathfrak{g}_2) + \mathfrak{g}_2 \Psi[\vartheta_1] - \mathfrak{g}_2 \varrho_2 \Phi_1(\varphi_1, \varphi_2) - \frac{\mathfrak{g}_2 \sigma}{\varrho_0} (\varphi_1 + \varphi_2) \right) - \mathfrak{g}_2 \Phi_1(\varphi_1, \varphi_2) + m_2 \mathfrak{g}_1 \varrho_0^{-1} \zeta_2, \\ b_{32} &= \varrho_0^{-1} \zeta_1 b_{12} - b_{22}, \quad b_{34} = \varrho_0^{-1} \Psi_1[\vartheta_2] + m_2 \mathfrak{g}_1 (1 + \varrho_0^{-1} \varphi_1), \\ b_{30} &= \varrho_0^{-1} \zeta_1 (b_{10} - m_1 \varrho_1) + \Psi[\vartheta_1] - m_2 (1 - \zeta_2 - \varrho_0^{-1} \zeta_2 \varphi_2), \\ \Phi_1(\varphi_1, \varphi_2) &= (p'(\varrho_0) \varrho_0)^{-1} \sigma \int_0^1 H'(\varphi_1 s + \varphi_2 (1 - s)) ds, \end{aligned}$$

and the operators  $\mathcal{C}_1$  and  $\mathcal{D}$  are defined by the equalities

$$\begin{aligned} \mathcal{C}(\mathbf{w}) &= \mathcal{B}_1(\psi, \mathbf{u}_1, \mathbf{u}_1) + \mathcal{B}_1(\varrho_2, \mathbf{w}, \mathbf{u}_1) + \mathcal{B}_1(\varrho_2, \mathbf{u}_2, \mathbf{w}), \\ \mathcal{D} &= \mathcal{A}_1(\mathbf{v}_2) - \mathcal{A}_2(\mathbf{v}_2) + R(\mathcal{B}_1(\varrho_2, \mathbf{u}_2, \mathbf{u}_2) - \mathcal{B}_2(\varrho_2, \mathbf{u}_2, \mathbf{u}_2)), \end{aligned}$$

where  $\mathcal{A}_i$  and  $\mathcal{B}_i$  are given by (1.8) with  $\mathbf{N}_i$  instead of  $\mathbf{N}$ .

We consider  $\mathcal{D}$  and  $\mathfrak{d}$  as *given functions*, and equality (2.17) as the system of equations and boundary conditions for unknowns  $\mathbf{w}$ ,  $\psi$ ,  $\xi$ , and  $n$ . The next step is crucial for further analysis. We replace equations (2.17) by an integral identity, which leads to the notion of a very weak solution of problem (2.17). To this end choose an arbitrary functions  $(\mathbf{H}, G, F, M) \in C^\infty(\Omega)^6$  such that  $G - \Pi G = 0$ , and consider the auxiliary boundary value problems

$$(2.19) \quad \mathcal{L}^* \varsigma = F, \quad \mathcal{L} v = M \text{ in } \Omega, \quad \varsigma = 0 \text{ on } \Sigma_{\text{out}}, \quad v = 0 \text{ on } \Sigma_{\text{in}}.$$

$$(2.20) \quad \Delta \mathbf{h} - \nabla g = \mathbf{H}, \quad \operatorname{div} \mathbf{h} = \Pi G \text{ in } \Omega, \quad \mathbf{h} = 0 \text{ on } \partial\Omega, \quad \Pi g = g.$$

Since, under the assumptions of Theorem 1.5,  $\mathbf{u}$  and  $\sigma$  meet all requirements of Corollary 1.4, each of problems (2.19) has a unique solution, such that

$$(2.21) \quad \|\varsigma\|_{H^{s,r}(\Omega)} \leq c \|F\|_{H^{s,r}(\Omega)}, \quad \|v\|_{H^{s,r}(\Omega)} \leq c \|M\|_{H^{s,r}(\Omega)},$$

where  $c$  depends only on  $\mathbf{U}$ ,  $\Omega$ ,  $r$ ,  $s$ , and  $\sigma$ . On the other hand, by virtue of Lemma 1.2, problem (2.20) has the unique solution satisfying the inequality

$$(2.22) \quad \|\mathbf{h}\|_{H^{1+s,r}(\Omega)} + \|g\|_{H^{s,r}(\Omega)} \leq c \|\mathbf{H}\|_{H^{1+s,r}(\Omega)} + c \|G\|_{H^{s,r}(\Omega)}.$$

Recall that  $\mathbf{w} \in H^{2,2}(\Omega)^3 \cap C^1(\Omega)^3$  vanishes on  $\partial\Omega$ , and  $(\omega, \psi, \xi) \in H^{1,2}(\Omega)^3 \cap C(\Omega)^3$ . Multiplying both sides of the first equation in system (2.17) by  $\varsigma$ , both sides of the fourth equation in (2.17) by  $v$ , integrating the results over  $\Omega$  and using the Green formula for the Stokes equations we obtain the system of integral equalities

$$(2.23) \quad \begin{aligned} \int_{\Omega} \psi F \, dx &= \int_{\Omega} (-\mathbf{w} \cdot \nabla \varphi_2 + b_{11}\psi + b_{12}\omega + b_{13}n + b_{10}\mathfrak{d}) \varsigma \, dx, \\ \int_{\Omega} \mathbf{w} \mathbf{H} \, dx + \int_{\Omega} \omega G \, dx &= \int_{\Omega} (b_{21}\psi + b_{22}\omega + b_{23}n + b_{20}\mathfrak{d}) g \, dx + \\ \int_{\Omega} (\mathcal{A}_1 \mathbf{w} + R\mathcal{C}(\mathbf{w}, \psi) + \mathcal{D}) \mathbf{h} \, dx, \quad \int_{\Omega} \xi M \, dx &= \int_{\Omega} (\operatorname{div}(\zeta_2 \mathbf{w}) + \sigma \mathfrak{d}) v \, dx. \end{aligned}$$

Next, since  $\operatorname{div}(\varrho_2 \mathbf{u}_2) = 0$ , we have

$$\begin{aligned} \int_{\Omega} (\mathcal{B}_1(\varrho_2, \mathbf{w}, \mathbf{u}_1) + \mathcal{B}_1(\varrho_2, \mathbf{u}_2, \mathbf{w})) \cdot \mathbf{h} \, dx &= \\ \int_{\Omega} \varrho_2 \mathbf{w} \cdot \left( \nabla(\mathbf{N}_1^{-1} \mathbf{u}_1) \cdot (\mathbf{N}_1^{-1} \mathbf{h}) - (\mathbf{N}_1^{-1})^* \nabla(\mathbf{N}_1^{-1} \mathbf{h})^* \mathbf{u}_2 \right) \, dx. \end{aligned}$$

On the other hand, integration by parts gives

$$\int_{\Omega} \operatorname{div}(\zeta_2 \mathbf{w}) v \, dx = - \int_{\Omega} \zeta_2 \mathbf{w} \nabla v \, dx.$$

Using these identities and recalling the duality pairing we can collect relations (2.23), together with the expression for  $n$ , in one integral identity

$$\begin{aligned}
 (2.24) \quad & \int_{\Omega} \mathbf{w} \left( \mathbf{H} - R\varrho_2 \nabla(\mathbf{N}_1^{-1} \mathbf{u}_1) \cdot (\mathbf{N}_1^{-1} \mathbf{h}) + R\varrho_2 (\mathbf{N}_1^{-1})^* (\nabla(\mathbf{N}_1^{-1} \mathbf{h})^* \mathbf{u}_2) \right) dx - \\
 & \mathfrak{B}(\mathbf{w}, \varphi_2, \varsigma) - \mathfrak{B}(\mathbf{w}, v, \zeta_2) - \mathfrak{A}_1(\mathbf{w}, \mathbf{h}) + \langle \omega, G - b_{12}\varsigma - b_{22}g - \varkappa b_{32} \rangle + \\
 & \langle \psi, F - b_{11}\varsigma - b_{21}g - \varkappa b_{31} - R\mathbf{u}_1 \cdot \nabla(\mathbf{N}_1^{-1} \mathbf{u}_1) \cdot \mathbf{N}_1^{-1} \mathbf{h} \rangle + \\
 & \langle \xi, M - \varkappa b_{34} \rangle + n - n \langle 1, b_{13}\varsigma + b_{23}g \rangle = \langle \mathfrak{d}, b_{10}\varsigma + b_{20}g + \varkappa b_{30} + \sigma v \rangle + \langle \mathcal{D}, \mathbf{h} \rangle.
 \end{aligned}$$

Here, the trilinear form  $\mathfrak{B}$  and the bilinear form  $\mathfrak{A}_1$  are defined by the equalities

$$\mathfrak{B}(\mathbf{w}, \varphi_2, \varsigma) = - \int_{\Omega} \varsigma \mathbf{w} \cdot \nabla \varphi_2 \, dx, \quad \mathfrak{A}_1(\mathbf{w}, \mathbf{h}) = \int_{\Omega} \mathcal{A}_1 \mathbf{w} \cdot \mathbf{h} \, dx.$$

Note, that relations (2.24) are well-defined for all  $\mathbf{w} \in \mathcal{H}_0^{1-s, r'}(\Omega)$  and  $\psi, \xi \in \mathbb{H}^{-s, r'}(\Omega)$ . It is obviously true for all terms, possibly except of  $\mathfrak{A}_1$  and  $\mathfrak{B}$ . Well-posedness of the form  $\mathfrak{B}$  follows from Lemma 1.6. The well posedness of the forms  $\mathfrak{A}_1$  results from the following lemma, the proof is given in Appendix A.

**Lemma 2.2.** *Let  $sr > 3$ ,  $1/2 \leq s \leq 1$  and  $\mathbf{w} \in \mathcal{H}_0^{1-s, r'}(\Omega) \cap H_0^{1,2}(\Omega)$ ,  $\mathbf{h} \in H^{1+s, r}(\Omega)$  and  $\mathbf{N}$  satisfy (1.38). Then there is a constant  $c$  depending only on  $s, r$  and  $\Omega$  such that*

$$(2.25) \quad |\mathfrak{A}(\mathbf{w}, \mathbf{h})| \leq c\tau^2 \|\mathbf{w}\|_{\mathcal{H}_0^{1-s, r'}(\Omega)} \|\mathbf{h}\|_{H^{1+s, r}(\Omega)}.$$

Hence the form  $\mathfrak{A}$  can be continuously extended to  $\mathfrak{A} : \mathcal{H}_0^{1-s, r'}(\Omega)^3 \times H^{1+s, r}(\Omega)^3 \mapsto \mathbb{R}$ .

Thus, relations (2.24) are well-defined for all  $(\mathbf{w}, \psi, \omega, \xi) \in \mathcal{H}_0^{1-s, r'}(\Omega)^3 \times \mathbb{H}^{-s, r'}(\Omega)^3$ . Equalities (2.24) along with equations (2.19), (2.20) are called *the very weak formulation* of problem (2.17). The natural question is the uniqueness of solutions to such weak formulation. The following theorem, which is the main result of this section, guarantees the uniqueness of very weak solutions for sufficiently small  $\tau$ .

**Theorem 2.3.** *Let  $s, r$ , and  $\sigma$  satisfy condition (1.37), parameters  $\lambda, R$ , matrices  $\mathbf{N}_i$ ,  $i = 1, 2$ , satisfy conditions (1.38), constants  $\tau$  meet all requirements of Theorem 1.5 and the solutions  $(\vartheta_i, \zeta_i, m_i)$ ,  $i = 1, 2$ , to problem (1.13) with the matrices  $\mathbf{N}_i$ ,  $i = 1, 2$ , belong to  $\mathcal{B}_{\tau} \times X^{s, r} \times \mathbb{R}$ . Furthermore, assume that for any  $(\mathbf{H}, G, F, M) \in C^\infty(\Omega)^6$  and for  $(\varsigma, v, \mathbf{h}, g)$  satisfying (2.19)-(2.20), the elements  $(\mathbf{w}, \omega, \psi, \xi) \in \mathcal{H}_0^{1-s, r'}(\Omega)^3 \times \mathbb{H}^{-s, r'}(\Omega)^3$  and the constant  $n$  satisfy identity (2.24).*

*Then, there are constants  $c, \tau_1$  depending only on  $s, r, \sigma$  and  $\Omega, \mathbf{U}$ , such that for  $\tau \in (0, \tau_1]$ , we have*

$$\begin{aligned}
 (2.26) \quad & \|\mathbf{w}\|_{\mathcal{H}_0^{1-s, r'}(\Omega)} + \|\omega\|_{\mathbb{H}^{-s, r'}(\Omega)} + \|\psi\|_{\mathbb{H}^{-s, r'}(\Omega)} + \|\xi\|_{\mathbb{H}^{-s, r'}(\Omega)} + |n| \leq \\
 & \leq c(\|\mathcal{D}\|_{L^1(\Omega)} + \|\mathfrak{d}\|_{\mathbb{H}^{-s, r'}(\Omega)}).
 \end{aligned}$$

*Proof.* The proof is based upon two auxiliary lemmas, the first lemma establishes the bounds for coefficients of problem (2.17).

**Lemma 2.4.** *Under the assumptions of Theorem 2.3, all the coefficients of identity (2.24) satisfy the inequalities  $\|b_{ij}\|_{X^{s,r}} \leq c$ , furthermore*

$$(2.27) \quad \begin{aligned} \|b_{12}\|_{X^{s,r}} + \|b_{22}\|_{X^{s,r}} + \|b_{11}\|_{X^{s,r}} + \|b_{10}\|_{X^{s,r}} + \|b_{20}\|_{X^{s,r}} &\leq c\tau, \\ \|b_{31}\|_{X^{s,r}} + \|b_{32}\|_{X^{s,r}} + \|b\|_{X^{s,r}} &\leq c\tau. \end{aligned}$$

*Proof.* The proof follows from Lemma 2.1 combined with formulae (2.18).  $\square$

In order to formulate the second auxiliary result we introduce the following denotations.

$$\begin{aligned} \mathfrak{I}_1 &= \langle \psi, b_{11}\varsigma \rangle + \langle \omega, b_{12}\varsigma \rangle + \langle \mathfrak{d}, b_{10}\varsigma \rangle, \quad \mathfrak{I}_2 = \langle \psi, b_{21}g \rangle + \langle \omega, b_{22}g \rangle + \langle \mathfrak{d}, b_{20}g \rangle, \\ \mathfrak{I}_3 &= \kappa(\langle \psi, b_{31} \rangle + \langle \omega, b_{32} \rangle + \langle \xi, b_{34} \rangle + \langle \mathfrak{d}, b_{30} \rangle), \quad \mathfrak{I}_4 = \langle \psi, \mathbf{u}_1 \nabla(\mathbf{N}_1 \mathbf{u}_1) \cdot \mathbf{N}_1^{-1} \mathbf{h} \rangle, \\ \mathfrak{I}_5 &= \int_{\Omega} \varrho_2 \mathbf{w} \cdot \left( \nabla(\mathbf{N}_1^{-1} \mathbf{u}_1) \cdot (\mathbf{N}_1^{-1} \mathbf{h}) - (\mathbf{N}_1^{-1})^* \nabla(\mathbf{N}_1^{-1} \mathbf{h})^* \mathbf{u}_2 \right) dx, \\ \mathfrak{G} &= \|\mathbf{w}\|_{\mathcal{H}_0^{1-s,r'}(\Omega)} + \|\psi\|_{\mathbb{H}^{-s,r'}(\Omega)} + \|\omega\|_{\mathbb{H}^{-s,r'}(\Omega)} + \|\xi\|_{\mathbb{H}^{-s,r'}(\Omega)}, \\ \mathfrak{Q} &= \|\mathbf{H}\|_{\mathcal{H}^{s-1,r}(\Omega)} + \|G\|_{H^{s,r}(\Omega)} + \|F\|_{H^{s,r}(\Omega)} + \|M\|_{H^{s,r}(\Omega)}. \end{aligned}$$

**Lemma 2.5.** *Under the assumptions of Theorem 2.3, there is a constant  $c$ , depending only on  $\mathbf{U}$ ,  $\Omega$ ,  $s$ ,  $r$ , and  $\sigma$ , such that*

$$(2.28) \quad \mathfrak{I}_1 \leq c\tau \mathfrak{Q} [\mathfrak{G} + \|\mathfrak{d}\|_{\mathbb{H}^{-s,r'}(\Omega)}]$$

$$(2.29) \quad \mathfrak{I}_2 \leq c\mathfrak{Q} [\tau \mathfrak{G} + \|\psi\|_{\mathbb{H}^{-s,r'}(\Omega)} + \|\mathfrak{d}\|_{\mathbb{H}^{-s,r'}(\Omega)}]$$

$$(2.30) \quad \mathfrak{I}_3 \leq c\tau [\mathfrak{G} + \|\mathfrak{d}\|_{\mathbb{H}^{-s,r'}(\Omega)}], \quad \mathfrak{I}_4 + \mathfrak{I}_5 \leq c\mathfrak{Q}\mathfrak{G}$$

*Proof.* We have

$$\begin{aligned} \langle \psi, b_{11}\varsigma \rangle + \langle \omega, b_{12}\varsigma \rangle + \langle \mathfrak{d}, b_{10}\varsigma \rangle &\leq \|b_{11}\varsigma\|_{H^{s,r}(\Omega)} \|\psi\|_{\mathbb{H}^{-s,r'}(\Omega)} + \\ &\quad \|b_{12}\varsigma\|_{H^{s,r}(\Omega)} \|\omega\|_{\mathbb{H}^{-s,r'}(\Omega)} + \|b_{10}\varsigma\|_{H^{s,r}(\Omega)} \|\mathfrak{d}\|_{\mathbb{H}^{-s,r'}(\Omega)}, \end{aligned}$$

Recall that for  $rs > 3$ ,  $H^{s,r}(\Omega)$  is a Banach algebra. From this, estimate (2.21) and inequalities (2.27) we obtain

$$\begin{aligned} &\|b_{11}\varsigma\|_{H^{s,r}(\Omega)} \|\psi\|_{\mathbb{H}^{-s,r'}(\Omega)} + \|b_{12}\varsigma\|_{H^{s,r}(\Omega)} \|\omega\|_{\mathbb{H}^{-s,r'}(\Omega)} + \|b_{10}\varsigma\|_{H^{s,r}(\Omega)} \|\mathfrak{d}\|_{\mathbb{H}^{-s,r'}(\Omega)} \leq \\ &c\|\varsigma\|_{H^{s,r}(\Omega)} (\|b_{11}\|_{H^{s,r}(\Omega)} \|\psi\|_{\mathbb{H}^{-s,r'}(\Omega)} + \|b_{12}\|_{H^{s,r}(\Omega)} \|\omega\|_{\mathbb{H}^{-s,r'}(\Omega)} + \|b_{10}\|_{H^{s,r}(\Omega)} \|\mathfrak{d}\|_{\mathbb{H}^{-s,r'}(\Omega)}) \\ &\leq c\tau \|F\|_{H^{s,r}(\Omega)} (\|\psi\|_{\mathbb{H}^{-s,r'}(\Omega)} + \|\omega\|_{\mathbb{H}^{-s,r'}(\Omega)} + \|\mathfrak{d}\|_{\mathbb{H}^{-s,r'}(\Omega)}), \end{aligned}$$

which gives (2.28). Repeating these arguments and using inequality (2.21) we obtain the estimates for  $\mathfrak{I}_2$  and  $\mathfrak{I}_3$ . Next, we have

$$\|\mathbf{u}_1 \cdot \nabla(\mathbf{N}_1^{-1} \mathbf{u}_1) \cdot \mathbf{N}_1^{-1} \mathbf{h}\|_{H^{s,r}(\Omega)} \leq c\|\mathbf{u}_1\|_{H^{s,r}(\Omega)} \|\mathbf{u}_1\|_{H^{1+s,r}(\Omega)} \|\mathbf{h}\|_{H^{1+s,r}(\Omega)} \leq c\|\mathbf{H}\|_{H^{1+s,r}(\Omega)}$$

which gives the estimate for  $\mathfrak{I}_4$ . Since the embeddings  $H^{s,r}(\Omega) \hookrightarrow C(\Omega)$ ,  $H^{1+s,r}(\Omega) \hookrightarrow C^1(\Omega)$  are bounded and  $\|\mathbf{N}_1^{\pm 1}\|_{C^1(\Omega)} \leq c$ , we have

$$\varrho_2 |\nabla(\mathbf{N}_1^{-1} \mathbf{u}_1)| |\mathbf{N}_1^{-1} \mathbf{h}| + \varrho_2 |(\mathbf{N}_1^{-1} \mathbf{u}_2)| |\nabla(\mathbf{N}_1^{-1} \mathbf{h})| \leq c\|\mathbf{h}\|_{H^{1+s,r}(\Omega)},$$

which leads to the inequality

$$\mathfrak{I}_5 \leq c\|\mathbf{h}\|_{H^{1+s,r}(\Omega)} \|\mathbf{w}\|_{L^1(\Omega)} \leq c(\|\mathbf{H}\|_{\mathcal{H}^{s-1,r}(\Omega)} + \|G\|_{H^{s,r}(\Omega)}) \|\mathbf{w}\|_{\mathcal{H}^{1-s,r'}(\Omega)},$$

and the proof of Lemma 2.5 is completed.  $\square$

Let us return to the proof of Theorem 2.3. It follows from the duality principle that the theorem is proved provided we show that, under the assumptions of Theorem 1.5, the following inequality holds

$$(2.31) \quad \sup_{\Omega(\mathbf{H}, G, F, M)=1} (\langle \mathbf{w}, \mathbf{H} \rangle + \langle \omega, G \rangle + \langle \psi, F \rangle + \langle \xi, M \rangle) + |n| \leq c\tau (\mathfrak{G}(\mathbf{w}, \omega, \psi, \xi) + |n|) + c(\|\mathcal{D}\|_{L^1(\Omega)} + \|\mathfrak{d}\|_{\mathbb{H}^{-s, r'}(\Omega)}),$$

where the constant  $c$  depends only on  $\Omega, \mathbf{U}$  and  $r, s, \sigma$ . Therefore, our task is to estimate step by step all terms in the left hand side of (2.31). We begin with an estimate for the term  $\langle \psi, F \rangle$ . To this end, take  $\mathbf{H} = \mathbf{h} = 0, G = g = 0, M = v = 0$ , and rewrite identity (2.24) in the form

$$\langle \psi, F \rangle = \mathfrak{B}(\mathbf{w}, \varphi_2, \varsigma) + \mathfrak{I}_1 + \mathfrak{I}_3 + n\langle 1, b_{13}\varsigma \rangle - n.$$

By virtue of Lemma 1.6 and estimate (2.22) we have

$$(2.32) \quad \mathfrak{B}(\mathbf{w}, \varphi_2, \varsigma) \leq c\tau \|\mathbf{w}\|_{\mathcal{H}_0^{1-s, r'}(\Omega)} \|\varsigma\|_{H^{s, r}(\Omega)} \leq c\tau \|\mathbf{w}\|_{H^{1-s, r'}(\Omega)} \|F\|_{H^{s, r}(\Omega)}.$$

On the other hand, Lemma 2.4 and inequality (2.21) yield  $|\langle 1, b_{13}\varsigma \rangle| \leq c\|F\|_{H^{s, r}(\Omega)}$ . From this and (2.28), (2.30) we finally obtain

$$(2.33) \quad \langle \psi, F \rangle \leq |n| + c\|F\|_{H^{s, r}(\Omega)} [\tau\mathfrak{G} + \|\mathfrak{d}\|_{\mathbb{H}^{-s, r'}(\Omega)} + |n|].$$

Moreover, by virtue of the duality principle

$$\|\psi\|_{\mathbb{H}^{-s, r'}(\Omega)} = \sup_{\|F\|_{H^{s, r}(\Omega)}=1} |\langle \psi, F \rangle|,$$

we have the following estimate for  $\psi$

$$(2.34) \quad \|\psi\|_{\mathbb{H}^{-s, r'}(\Omega)} \leq c\tau\mathfrak{G} + c\|\mathfrak{d}\|_{\mathbb{H}^{-s, r'}(\Omega)} + c|n|.$$

Let us estimate  $\mathbf{w}$  and  $\omega$ . Substituting  $F = \varsigma = 0$  and  $M = v = 0$  into (2.24) we obtain

$$\langle \mathbf{w}, \mathbf{H} \rangle + \langle \omega, G \rangle = \mathfrak{A}_1(\mathbf{w}, \mathbf{h}) + \mathfrak{I}_2 + \mathfrak{I}_3 + R\mathfrak{I}_4 + R\mathfrak{I}_5 + n\langle 1, b_{23}g \rangle - n + \langle \mathcal{D}, \mathbf{h} \rangle$$

By virtue of Lemma 2.2 and (2.22), the first term in the right hand side is bounded

$$|\mathfrak{A}_1(\mathbf{w}, \mathbf{h})| \leq c\tau^2 (\|\mathbf{H}\|_{\mathcal{H}^{s-1, r}(\Omega)} + \|G\|_{H^{s, r}(\Omega)}) \|\mathbf{w}\|_{\mathcal{H}_0^{1-s, r'}(\Omega)}.$$

Next we have

$$|n\langle 1, b_{23}g \rangle| \leq c(\|\mathbf{H}\|_{\mathcal{H}^{s-1, r}(\Omega)} + \|G\|_{H^{s, r}(\Omega)})|n|.$$

Obviously

$$|\langle \mathcal{D}, \mathbf{h} \rangle| \leq c\|\mathcal{D}\|_{L^1(\Omega)} \|\mathbf{h}\|_{C(\Omega)} \leq c(\|\mathbf{H}\|_{\mathcal{H}^{s-1, r}(\Omega)} + \|G\|_{H^{s, r}(\Omega)}) \|\mathcal{D}\|_{L^1(\Omega)}.$$

These inequalities together with estimates (2.29)-(2.30) and inequality  $R \leq \tau^2$  imply

$$\begin{aligned} \langle \mathbf{w}, \mathbf{H} \rangle + \langle \omega, G \rangle &\leq |n| + c\tau\mathfrak{Q}\mathfrak{G} + \\ &c\mathfrak{Q}(\|\psi\|_{\mathbb{H}^{-s, r'}(\Omega)} + |n| + \|\mathfrak{d}\|_{\mathbb{H}^{-s, r'}(\Omega)} + \|\mathcal{D}\|_{L^1(\Omega)}). \end{aligned}$$

Combining this result with (2.34) we obtain

$$(2.35) \quad \langle \mathbf{w}, \mathbf{H} \rangle + \langle \omega, G \rangle \leq |n| + c\tau\mathfrak{Q}\mathfrak{G} + c\mathfrak{Q}(|n| + \|\mathfrak{d}\|_{\mathbb{H}^{-s, r'}(\Omega)} + \|\mathcal{D}\|_{L^1(\Omega)}),$$



where  $\mathfrak{Q} = \mathfrak{Q}(\mathbf{H}, G, 0, 0)$ . For  $G = 0$  and by the duality principle

$$\|\mathbf{w}\|_{\mathcal{H}_0^{1-s, r'}(\Omega)} = \sup_{\|\mathbf{H}\|_{\mathcal{H}^{s-1, r}(\Omega)}=1} \langle \mathbf{H}, \mathbf{w} \rangle,$$

we conclude from this that

$$(2.36) \quad \|\mathbf{w}\|_{\mathcal{H}_0^{1-s, r'}(\Omega)} \leq |n| + c\tau\mathfrak{G} + c\left(|n| + \|\mathfrak{d}\|_{\mathbb{H}^{-s, r'}(\Omega)} + \|\mathcal{D}\|_{L^1(\Omega)}\right).$$

Next, substituting  $\mathbf{H} = \mathbf{h} = 0$ ,  $G = g = F = \varsigma = 0$  into identity (2.24), we arrive at

$$\langle \xi, M \rangle = \mathfrak{B}(\mathbf{w}, \zeta_2, v) + \mathfrak{I}_3 + \sigma\langle \mathfrak{d}, v \rangle - n.$$

Lemma 1.6 and (2.21) give the estimate for the first term

$$|\mathfrak{B}(\mathbf{w}, \zeta_2, v)| \leq c\|\mathbf{w}\|_{\mathcal{H}_0^{1-s, r'}(\Omega)}\|v\|_{H^{s, r}(\Omega)} \leq c\|\mathbf{w}\|_{\mathcal{H}_0^{1-s, r'}(\Omega)}\|M\|_{H^{s, r}(\Omega)}.$$

From this and estimates (2.30), (2.21), we obtain

$$\langle \xi, M \rangle \leq c\tau\mathfrak{Q}\mathfrak{G} + c\mathfrak{Q}(\|\mathbf{w}\|_{\mathcal{H}_0^{1-s, r'}(\Omega)} + \|\mathfrak{d}\|_{\mathbb{H}^{-s, r'}(\Omega)}) + |n|.$$

Combining this result with inequality (2.36) we arrive at

$$(2.37) \quad \langle \xi, M \rangle \leq c\mathfrak{Q}(\tau\mathfrak{G} + |n| + \|\mathfrak{d}\|_{\mathbb{H}^{-s, r'}(\Omega)} + \|\mathcal{D}\|_{L^1(\Omega)}) + c|n|.$$

Finally, choosing all test functions in (2.24) equal to 0 we obtain  $n = \mathfrak{I}_3$  which together with (2.30) yields

$$(2.38) \quad |n| \leq c\tau\mathfrak{Q}(\mathfrak{G} + \|\mathfrak{d}\|_{\mathbb{H}^{-s, r'}(\Omega)}).$$

From (2.33), (2.35), (2.37), combined with (2.38), it follows (2.31) and the proof of Theorem 2.3 is completed.  $\square$

*Uniqueness of solutions.* The important consequence of Theorem 2.3 is the following result on uniqueness of solutions to problem (1.13).

**Proposition 2.6.** *Under the assumptions of Theorem 1.5, there exists positive  $\tau_0$  such that for all  $\tau \in (0, \tau_0)$ , problem (1.13) admits a unique solution in the ball  $\mathcal{B}_\tau$ .*

*Proof.* If for some  $\mathbf{N}$ , the problem has two distinct solutions  $(\vartheta_i, \zeta_i, m_i)$ ,  $i = 1, 2$ , with  $\vartheta_i \in \mathcal{B}_\tau$ , then the corresponding *finite differences* of the solutions  $\mathbf{w}$ ,  $\psi$ ,  $\omega$  and  $\xi$  meet all requirements of Theorem 2.3 with  $\mathfrak{d} = 0$  and  $\mathcal{D} = 0$ . Therefore, in view of (2.26) all the elements  $\mathbf{w}$ ,  $\psi$ ,  $\omega$  and  $\xi$  are equal to 0, which completes the proof.  $\square$

### 3. PROOFS OF THEOREM 1.8 AND THEOREM 1.10

*Proof of Theorem 1.8.* Let us consider a family of matrices  $\mathbf{N}(\varepsilon)$  having representation (1.6) and the sequence of corresponding solutions  $(\vartheta(\varepsilon), \zeta(\varepsilon), m(\varepsilon))$  to problem (1.13), where  $\vartheta(\varepsilon) = (\mathbf{v}(\varepsilon), \pi(\varepsilon), \varphi(\varepsilon))$ . By virtue of (1.40), we can assume that, possibly after passing to a subsequence, the sequence  $(\vartheta(\varepsilon), \zeta(\varepsilon), m(\varepsilon))$  converges weakly in  $(Y^{s, r})^3 \times (X^{s, r})^3 \times \mathbb{R}$  to some element  $(\vartheta, \zeta, m)$ , which satisfies equations (1.13) with  $\mathbf{N} = \mathbf{I}$  and meets all requirements of Theorem 1.5. Since the solution  $(\vartheta, \zeta, m)$  to problem (1.13) is unique, the limit is independent on the choice of a subsequence and the whole sequence converges to the limit  $(\vartheta, \zeta, m)$ . It follows from (2.17) that the differences  $\vartheta - \vartheta(\varepsilon)$ ,  $\zeta - \zeta(\varepsilon)$ ,  $m - m(\varepsilon)$ , satisfy equations (2.17), with the coefficients  $b_{ij}^{(\varepsilon)}$  and the operator  $\mathcal{D}_\varepsilon$  given by formulae (2.18) with

$$\mathbf{N}_1 = \mathbf{I}, \mathbf{N}_2 = \mathbf{N}(\varepsilon), (\vartheta_1, \zeta_1, m_1) = (\vartheta, \zeta, m), (\vartheta_2, \zeta_2, m_2) = (\vartheta(\varepsilon), \zeta(\varepsilon), m(\varepsilon)).$$

In particular, the operator  $\mathcal{D}_\varepsilon$  is defined by the equality

$$\begin{aligned} \mathcal{D}_\varepsilon = & R\varrho(\varepsilon) \left( \mathbf{u}(\varepsilon) \nabla \mathbf{u}(\varepsilon) - (\mathbf{N}(\varepsilon)^*)^{-1} (\mathbf{u}(\varepsilon) \nabla (\mathbf{N}(\varepsilon)^{-1} \mathbf{u}(\varepsilon))) \right) + \\ & \mathbf{N}(\varepsilon)^{-1} \operatorname{div} \left( \mathbf{g}^{-1}(\varepsilon) \mathbf{N}(\varepsilon) \mathbf{N}(\varepsilon)^* \nabla (\mathbf{N}(\varepsilon)^{-1} \mathbf{u}(\varepsilon)) \right) - \Delta \mathbf{u}(\varepsilon) \end{aligned}$$

and admits the representation  $\mathcal{D}_\varepsilon = \varepsilon \mathcal{D}_0(b, \mathbf{D}) + \varepsilon^2 \mathcal{D}_1(\varepsilon)$ , where  $\mathcal{D}_0$  is given by (1.44). Moreover, since the norms  $\|\varrho(\varepsilon)\|_{C(\Omega)}$  and  $\|\mathbf{u}(\varepsilon)\|_{H^{2,2}(\Omega)}$  are uniformly bounded, we have

$$(3.1) \quad \|\mathcal{D}_1(\varepsilon)\|_{L^2(\Omega)} \leq c(\mathbf{U}, \Omega, \sigma).$$

Next, note that  $\mathbf{g}(\varepsilon)$  admits the decomposition  $\mathbf{g}(\varepsilon) = 1 + \varepsilon \mathfrak{d} + \varepsilon^2 \mathfrak{d}_1(\varepsilon)$ , where  $\mathfrak{d} = \operatorname{Tr} \mathbf{D}$ , and the reminder  $\mathfrak{d}_1(\varepsilon)$  is uniformly bounded in  $C^1(\Omega)$ . Proceeding as in the previous section and recalling the equalities  $\mathcal{A}_1 = \mathfrak{A}_1 = 0$ , we conclude that the finite differences

$$(\mathbf{w}_\varepsilon, \omega_\varepsilon, \psi_\varepsilon) = \varepsilon^{-1}(\vartheta - \vartheta(\varepsilon)), \quad \xi_\varepsilon = \varepsilon^{-1}(\zeta - \zeta(\varepsilon)), \quad n_\varepsilon = \varepsilon^{-1}(m - m(\varepsilon))$$

satisfy the integral identity

$$\begin{aligned} (3.2) \quad & \int_{\Omega} \mathbf{w}_\varepsilon \left( \mathbf{H} - \varrho(\varepsilon) \nabla \mathbf{u} \cdot \mathbf{h} + \varrho(\varepsilon) \nabla \mathbf{h}^* \mathbf{u}(\varepsilon) \right) dx - \\ & \mathfrak{B}(\mathbf{w}_\varepsilon, \varphi(\varepsilon), \varsigma) - \mathfrak{B}(\mathbf{w}_\varepsilon, v, \zeta(\varepsilon)) + \langle \omega_\varepsilon, G - b_{12}^{(\varepsilon)} \varsigma - b_{22}^{(\varepsilon)} g - \kappa b_{32}^{(\varepsilon)} \rangle + \\ & \langle \psi_\varepsilon, F - b_{11}^{(\varepsilon)} \varsigma - b_{21}^{(\varepsilon)} g - \kappa b_{31}^{(\varepsilon)} - R \mathbf{u} \cdot \nabla \mathbf{u} \cdot \mathbf{h} \rangle + \langle \xi_\varepsilon, M - \kappa b_{34}^{(\varepsilon)} \rangle + n_\varepsilon - \\ & n_\varepsilon \langle 1, b_{13}^{(\varepsilon)} \varsigma + b_{23}^{(\varepsilon)} g \rangle = \langle \mathfrak{d} + \varepsilon \mathfrak{d}_1(\varepsilon), b_{10}^{(\varepsilon)} \varsigma + b_{20}^{(\varepsilon)} g + \kappa b_{30}^{(\varepsilon)} + \sigma v \rangle + \langle \mathcal{D}_0 + \varepsilon \mathcal{D}_1(\varepsilon), \mathbf{h} \rangle. \end{aligned}$$

along with the orthogonality conditions  $\langle \omega_\varepsilon, 1 \rangle = 0$ . Here  $(\mathbf{H}, G, F, M)$  are arbitrary smooth functions such that  $G = \Pi G$ , the test functions  $\varsigma, v, g$  and  $\mathbf{h}$  are defined by equations (2.19), (2.20), and are independent on  $\varepsilon$ . Recall that the elements  $\vartheta_1 = \vartheta$  and  $\vartheta_2 = \vartheta(\varepsilon)$  belong to the ball  $\mathcal{B}_\tau$  and meet all requirements of Theorem 2.3. Hence, there exist  $\tau_1 > 0$ , depending only on  $\Omega, \mathbf{U}$  and  $s, r, \sigma$ , such that the conditions

$$\lambda^{-1}, R \leq \tau^2, \quad \|\mathbf{N}(\varepsilon) - \mathbf{I}\|_{C^1(\Omega)} \leq \tau^2, \quad 0 < \tau \leq \tau_1,$$

imply

$$\begin{aligned} \|\mathbf{w}_\varepsilon\|_{\mathcal{H}_0^{1-s, r'}(\Omega)} + \|\omega_\varepsilon\|_{\mathbb{H}^{-s, r'}(\Omega)} + \|\psi_\varepsilon\|_{\mathbb{H}^{-s, r'}(\Omega)} + \|\xi_\varepsilon\|_{\mathbb{H}^{-s, r'}(\Omega)} + |n_\varepsilon| \leq \\ c(\|\mathcal{D}_0 + \varepsilon \mathcal{D}_1\|_{L^1(\Omega)} + \|b + \varepsilon b_1\|_{C^2(\Omega)}) \leq c. \end{aligned}$$

Therefore, after possibly passing to a subsequence, we can assume that the sequence  $\mathbf{w}_\varepsilon$  converges to  $\mathbf{w}$  weakly in  $\mathcal{H}_0^{1-s, r'}(\Omega)$ , and  $(\omega_\varepsilon, \psi_\varepsilon, \xi_\varepsilon)$  converge to  $(\omega, \psi, \xi)$  (\*)-weakly in  $\mathbb{H}^{-s, r'}(\Omega)$  as  $\varepsilon \rightarrow 0$ .

Next, choose  $s' > s$  satisfying conditions (1.37). By virtue of Theorem 1.5, there exists  $\tau'_0 > 0$  (depending only on  $\Omega, \mathbf{U}, r, s, \sigma$ ) such that for all  $\tau \in (0, \tau'_0]$ , the functions  $(\vartheta(\varepsilon), \zeta(\varepsilon))$  are bounded in  $(Y^{r, s'})^3 \times (X^{r, s'})^3$ . It follows from this that the family  $\mathbf{u}(\varepsilon) = \mathbf{u}_0 + \mathbf{v}(\varepsilon)$  converges to  $\mathbf{u}$  strongly in  $Y^{s, r}$ , and  $(\varphi(\varepsilon), \pi(\varepsilon), \zeta(\varepsilon))$  converges to  $(\varphi, \pi, \zeta)$  strongly in  $X^{s, r}$ . Therefore, by virtue of Lemma 2.1, the sequence  $b_{ij}^{(\varepsilon)}$  converges strongly in  $X^{s, r}$  to  $b_{ij}^0$ . Hence, we can pass to the limit in (3.2). It is easy to see that the limits, the vector field  $\mathbf{w}$  and the functionals  $\psi, \omega, \xi$ , are given by a unique weak solution to problem (1.41) and, in addition, meet all requirements of Definition 1.7. It remains to note that by virtue of Theorem 2.3,

the limit is independent of the choice of a subsequence, which completes the proof of Theorem 1.8.  $\square$

*Proof of Theorem 1.10.* Assume that  $r, s, \sigma$ , and  $\tau$  satisfy inequalities (1.30), (1.37), and  $\vartheta = (\mathbf{v}, \pi, \varphi) \in \mathcal{B}_\tau$  be a solution to problem (1.13) given by Theorem 1.5. Denote by  $Y_0^{s,r}$  the subspace of the space  $Y^{s,r}$  of all functions vanishing on  $\partial\Omega$ , by  $X_{\text{in}}^{s,r}$  and  $X_{\text{out}}^{s,r}$  the subspaces of  $X^{s,r}$  which consist of all functions vanishing on  $\Sigma_{\text{in}}$  and  $\Sigma_{\text{out}}$ , respectively, and by  $X_\Pi^{s,r}$  the subspace of all function in  $X^{s,r}$  having the zero mean value. Introduce the Banach spaces  $\mathcal{E} = (Y_0^{s,r})^3 \times X_\Pi^{s,r} \times X_{\text{out}}^{s,r} \times X_{\text{in}}^{s,r} \times \mathbb{R}$ , and  $\mathcal{F} = (Z^{s,r})^3 \times X_\Pi^{s,r} \times (X^{s,r})^2 \times \mathbb{R}$ . Our first task is to show that for all  $\Theta \in \mathcal{F}$ , problem (1.51), (1.48) has a unique solution  $Y \in \mathcal{E}$ . We begin with the observation that, by virtue of Lemma 1.25, the Stokes operator has the bounded inverse

$$\begin{pmatrix} \Delta & -\nabla \\ \operatorname{div} & 0 \end{pmatrix}^{-1} : (Z^{s,r})^3 \times X_\Pi^{s,r} \rightarrow (Y_0^{s,r})^3 \times X_\Pi^{s,r}.$$

On the other hand, by virtue of Corollary 1.4, the operators  $\mathcal{L}$  and  $\mathcal{L}^*$  (1.48) have the bounded inverses  $\mathcal{L}^{-1} : X^{s,r} \rightarrow X_{\text{in}}^{s,r}$ ,  $(\mathcal{L}^*)^{-1} : X^{s,r} \rightarrow X_{\text{out}}^{s,r}$ . Therefore, there exists the bounded operator

$$\begin{pmatrix} \mathcal{L}^* & 0 & 0 \\ 0 & \mathcal{L} & 0 \\ -\mathbb{B}_{1,3} & 0 & 1 \end{pmatrix}^{-1} : (X^{s,r})^2 \times \mathbb{R} \rightarrow X_{\text{out}}^{s,r} \times X_{\text{in}}^{s,r} \times \mathbb{R}.$$

It follows from this that for all  $\Theta \in \mathcal{F}$ , the equation  $\mathfrak{L}\mathbf{Y} = \Theta$  has a unique solution satisfying boundary conditions (1.48) and the inequality  $\|\mathbf{Y}\|_{\mathcal{E}} \leq c\|\Theta\|_{\mathcal{F}}$ , where the constant  $c$  is independent of  $\tau$ . Let us consider the operators  $\mathfrak{U}$ . By virtue of Lemma 1.6, we have

$$\|\varsigma \nabla \varphi\|_{\mathcal{H}^{s-1,r}(\Omega)} \leq c\tau \|\varsigma\|_{X^{s,r}}.$$

It is easy to see that

$$\|\varsigma \nabla \varphi\|_{L^2(\Omega)} \leq c\|\varsigma\|_{X^{s,r}} \|\nabla \varphi\|_{L^2(\Omega)} \leq c\|\varsigma\|_{X^{s,r}} \|\varphi\|_{X^{s,r}} \leq c\tau \|\varsigma\|_{X^{s,r}}.$$

Combining the obtained estimates we get the inequality  $\|\varsigma \nabla \varphi\|_{Z^{s,r}} \leq c\tau \|\varsigma\|_{X^{s,r}}$ . Repetition of these arguments gives the inequality  $\|\varsigma \nabla v\|_{Z^{s,r}} \leq c\|v\|_{X^{s,r}}$ . Since the norms  $\|b_{ij}^0\|_{X^{s,r}}$  are uniformly bounded, we conclude from this that  $\|\mathfrak{U}\mathbf{Y}\|_{\mathcal{F}} \leq c\|\mathbf{Y}\|_{\mathcal{E}}$ . Finally, let us consider the operator  $\mathfrak{V}$ . Since the space  $X^{s,r}$  is the commutative Banach algebra and  $\nabla \mathbf{u} \in X^{s,r}$ , we have

$$\|R\mathbf{u}\nabla \mathbf{u}\mathbf{h}\|_{X^{s,r}} \leq c\tau^2 \|\mathbf{h}\|_{X^{s,r}}, \quad \|R\varrho(\nabla \mathbf{u} + \mathbf{u}\nabla)\mathbf{h}\|_{X^{s,r}} \leq c\tau^2 \|\mathbf{h}\|_{Y^{s,r}}.$$

On the other hand, by virtue of Lemma 2.1 and (1.42), the coefficients  $b_{ij}^0$  in the expression for  $\mathfrak{V}$  satisfy the inequalities

$$\|b_{12}^0\|_{X^{s,r}} + \|b_{11}^0\|_{X^{s,r}} + \|b_{31}^0\|_{X^{s,r}} + \|b_{32}^0\|_{X^{s,r}} + \|b_{34}^0\|_{X^{s,r}} \leq c\tau,$$

which yield the estimate  $\|\mathfrak{V}\mathbf{Y}\|_{\mathcal{F}} \leq c\tau \|\mathbf{Y}\|_{\mathcal{E}}$ . Thus we get that the diagonal matrix operator  $\mathfrak{L}$  has the bounded inverse,  $\mathfrak{U}$  is the bounded upper triangular (with respect to  $\mathfrak{L}$ ) matrix operator, and  $\mathfrak{V}$  is the small bounded operator. Hence, for all sufficiently small  $\tau$  the operator  $\mathfrak{L} - \mathfrak{U} - \mathfrak{V} : \mathcal{E} \rightarrow \mathcal{F}$  has the bounded inverse, which implies the existence of adjoint state satisfying equations (1.51) and boundary conditions (1.48).

It remains to prove identity (1.52). Fix the adjoint state  $\mathbf{Y} = (\mathbf{h}, g, \varsigma, v, l)$ , and set

$$\mathbf{H} = \Delta \mathbf{h} - \nabla g, \quad G = \operatorname{div} \mathbf{h}, \quad F = \mathcal{L}^* \varsigma, \quad M = \mathcal{L} v.$$

It follows from (1.51) that

$$\begin{aligned}
 \mathbf{H} - R\varrho(\nabla \mathbf{u} - \mathbf{u}\nabla)\mathbf{h} + \varsigma\nabla\varphi + \zeta\nabla v &= \Delta\eta\mathbf{U}_\infty + R\varrho((\mathbf{u}\nabla\eta)\mathbf{U}_\infty + (\mathbf{u}\mathbf{U}_\infty)\nabla\eta), \\
 (3.3) \quad G - \Pi(b_{21}^0\varsigma + b_{22}^0g - \varkappa b_{32}^0l) &= \Pi(\nabla\eta\mathbf{U}_\infty), \\
 F - Ru\nabla\mathbf{u}\mathbf{h} - b_{12}^0g - b_{11}^0\varsigma - \varkappa b_{31}^0l &= (\mathbf{u}\nabla\eta)(\mathbf{u}\mathbf{U}_\infty), \quad M = \varkappa b_{34}^0l.
 \end{aligned}$$

By virtue of Theorem 1.8, the shape derivative  $(\mathbf{w}, \omega, \psi, \xi, n)$  satisfies integral identities (1.46). On the other hand  $(F, \mathbf{H}, G, M)$  together with the components of the adjoint state  $\mathbf{Y}$  can be regarded as a collection of test functions for this identity. Substituting these test functions into in (1.46), using equalities (3.3), and recalling the identity  $\langle \omega, 1 \rangle = 0$ , we obtain

$$\begin{aligned}
 L_u(\mathbf{w}, \omega, \psi) + \varkappa(l-1)(\langle \psi, b_{31}^0 \rangle + \langle \omega, b_{32}^0 \rangle + \langle \xi, b_{34}^0 \rangle) + \\
 n - n\langle 1, b_{13}^0\varsigma \rangle = \langle \mathfrak{d}, b_{10}^0\varsigma + b_{20}^0g + \varkappa b_{30}^0 + \sigma v \rangle + \langle \mathcal{D}_0, \mathbf{h} \rangle.
 \end{aligned}$$

The most last equation in (1.51) reads  $l = \langle 1, b_{13}^0\varsigma \rangle$ , which leads to

$$\begin{aligned}
 (3.4) \quad L_u(\mathbf{w}, \omega, \psi) + \varkappa(l-1)(\varkappa(\langle \psi, b_{31}^0 \rangle + \langle \omega, b_{32}^0 \rangle + \langle \xi, b_{34}^0 \rangle) - n) = \\
 \langle \mathfrak{d}, b_{10}^0\varsigma + b_{20}^0g + \varkappa b_{30}^0 + \sigma v \rangle + \langle \mathcal{D}_0, \mathbf{h} \rangle.
 \end{aligned}$$

Next note that identities (1.46) imply the following expression for the constant  $n$ ,

$$n = \varkappa(\langle \psi, b_{31}^0 \rangle + \langle \omega, b_{32}^0 \rangle + \langle \xi, b_{34}^0 \rangle + \langle \mathfrak{d}, b_{30}^0 \rangle).$$

Substituting this equality into (3.4) and noting that  $\mathfrak{d} = \text{Tr } \mathbf{D}$  we obtain (1.52), which completes the proof.  $\square$ .

#### 4. PROOF OF THEOREM 1.3

Our strategy is the following. First we show that in the vicinity of each point  $P \in \Sigma_{\text{in}} \cup \Gamma$  there exist normal coordinates  $(y_1, y_2, y_3)$  such that  $\mathbf{u}\nabla_x = \mathbf{e}_1\nabla_y$ . Hence problem of existence of solutions to transport equation in the neighborhood of  $\Sigma_{\text{in}} \cap \Gamma$  is reduced to boundary problem for the model equation  $\partial_{y_1}\varphi + \sigma\varphi = f$  in a parabolic domain. Next we prove that the boundary value problem for the model equations has unique solution in fractional Sobolev space, which leads to the existence and uniqueness of solutions in the neighborhood of the inlet set. Using the existence of local solution we reduce problem (1.27) to the problem for modified equation, which does not require the boundary data. Application of well-known results on solvability of elliptic-hyperbolic equations in the case  $\Gamma = \emptyset$  gives finally the existence and uniqueness of solutions to problems (1.27) and (1.28).

**Lemma 4.1.** *Assume that the  $C^2$ -manifold  $\Sigma = \partial B$  and the vector field  $\mathbf{U} \in C^2(\Sigma)^3$  satisfy conditions (H1)-(H3). Let  $\mathbf{u} \in C^1(\mathbb{R}^3)^3$  be a compactly supported vector field such that*

$$\mathbf{u} = \mathbf{U} \text{ on } \Sigma, \quad \mathbf{u} = 0 \text{ on } S,$$

and denote  $M = \|\mathbf{u}\|_{C^1(\mathbb{R}^3)}$ . Then there is a  $a > 0$ , depending only on  $M$  and  $\Sigma$ , with the properties:

(P1) *For any point  $P \in \Gamma$  there exists a mapping  $y \rightarrow \mathbf{x}(y)$  which takes diffeomorphically the cube  $Q_a = [-a, a]^3$  onto a neighborhood  $\mathcal{O}_P$  of  $P$  and satisfies the equations*

$$(4.1) \quad \partial_{y_1}\mathbf{x}(y) = \mathbf{u}(\mathbf{x}(y)) \text{ in } Q_a, \quad \mathbf{x}(0, y_2, 0) \in \Gamma \cap \mathcal{O}_P \text{ for } |y_2| \leq a,$$

and the inequalities

$$(4.2) \quad \|\mathbf{x}\|_{C^1(Q_a)} + \|\mathbf{x}^{-1}\|_{C^1(\mathcal{O}_P)} \leq C_M, \quad |\mathbf{x}(y)| \leq C_M|y|,$$

where  $C_M = 3(1 + M^{-1})(M^2 + C_\Gamma^2 + 2)^{1/2}$  and  $C_\Gamma$  is the constant in condition **(H2)**.

**(P2)** There is a  $C^1$  function  $\Phi(y_1, y_2)$  defined in the square  $[-a, a]^2$  such that  $\Phi(0, y_2) = 0$ , and

$$(4.3) \quad \mathbf{x}(\{y_3 = \Phi\}) = \Sigma \cap \mathcal{O}_P, \quad \mathbf{x}(\{y_3 > \Phi\}) = \Sigma \cap \mathcal{O}_P.$$

Moreover  $\Phi$  is strictly decreasing in  $y_1$  for  $y_1 < 0$ , is strictly increasing in  $y_1$  for  $y_1 > 0$ , and satisfies the inequalities

$$(4.4) \quad C^- y_1^2 \leq \Phi(y_1, y_2) \leq C^+ y_1^2,$$

where the constants  $C^- = |\mathbf{U}(P)|N^-/12$  and  $C^+ = 12|\mathbf{U}(P)|N^+$  depend only on  $\mathbf{U}$  and  $\Sigma$ , where  $N^\pm$  are defined in Condition **(H2)**.

**(P3)** Introduce the sets

$$\begin{aligned} \Sigma_{\text{in}}^y &= \{(y_2, y_3) : |y_2| \leq a, 0 < y_3 < \Phi(-a, y_2)\}, \\ \Sigma_{\text{out}}^y &= \{(y_2, y_3) : |y_2| \leq a, 0 < y_3 < \Phi(a, y_2)\}. \end{aligned}$$

For every  $(y_2, y_3) \in \Sigma_{\text{in}}^y$  (resp.  $(y_2, y_3) \in \Sigma_{\text{out}}^y$ ), the equation  $y_3 = \Phi(y_1, y_2)$  has the unique negative (resp. positive) solution  $y_1 = a^-(y_2, y_3)$ , (resp.  $y_1 = a^+(y_2, y_3)$ ) such that

$$(4.5) \quad |\partial_{y_j} a^\pm(y_2, y_3)| \leq C/\sqrt{y_3}, \quad |a^\pm(y_2, y_3) - a^\pm(y'_2, y'_3)| \leq C(|y_2 - y'_2| + |y_3 - y'_3|)^{1/2}$$

**(P4)** Denote by  $G_a \subset Q_a$  the domain

$$(4.6) \quad G_a = \{y \in Q_a : \Phi(y_1, y_2) < y_3 < \Phi(-a, y_2)\},$$

and by  $B_P(\rho)$  the ball  $|x - P| \leq \rho$ . Then we have the inclusions

$$(4.7) \quad B_P(\rho_c) \subset \mathbf{x}(G_a) \subset \mathcal{O}_P \subset B_P(R_c),$$

where the constants  $\rho_c = a^2 C_M^{-1} C^-$ ,  $R_c = a C_M$ .

*Proof.* We start with the proof of **(P1)**. Choose the Cartesian coordinate system  $(x_1, x_2, x_3)$  associated with the point  $P$  and satisfying Condition **(H1)**. Let us consider the Cauchy problem.

$$(4.8) \quad \partial_{y_1} \mathbf{x} = \mathbf{u}(\mathbf{x}(y)) \text{ in } Q_a, \quad \mathbf{x} \Big|_{y_1=0} = \mathbf{x}_0(y_2) + y_3 \mathbf{e}_3.$$

Here the function  $\mathbf{x}_0$  is given by condition **(H2)**. Without loss of generality we can assume that  $0 < a < k < 1$ . It follows from **(H1)** that for any such  $a$ , problem (4.8) has the unique solution of class  $C^1(Q_a)$ . Next note that, by virtue of condition **(H1)**, for  $y_1 = 0$ , we have

$$(4.9) \quad |\mathbf{x}(y)| \leq (C_\Gamma + 1)|y|, \quad |\mathbf{u}(\mathbf{x}(y)) - \mathbf{u}(0)| \leq M(C_\Gamma + 1)|y|.$$

Denote by  $\mathfrak{F}(y) = D_y \mathbf{x}(y)$ . The calculations show that

$$\mathfrak{F}_0 := \mathfrak{F}(y) \Big|_{y_1=0} = \begin{pmatrix} u_1 & \Upsilon'(y_2) & 0 \\ u_2 & 1 & 0 \\ u_3 & \partial_{y_2} F(\Upsilon(y_2), y_2) & 1 \end{pmatrix}$$

and

$$\mathfrak{F}(0) = \begin{pmatrix} U & \Upsilon'(0) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which along with (4.9) implies

$$(4.10) \quad \|\mathfrak{F}(0)^{\pm 1}\| \leq C_M/3, \quad \|\mathfrak{F}_0(y) - \mathfrak{F}(0)\| \leq ca.$$

Differentiation of (4.8) with respect to  $y$  leads to the ordinary differential equation for  $\mathfrak{F}$

$$\partial_{y_1} \mathfrak{F} = D_y \mathbf{u}(\mathbf{x}) \mathfrak{F}, \quad \mathfrak{F}|_{y_1=0} = \mathfrak{F}_0.$$

Noting that  $\partial_{y_1} \|\mathfrak{F} - \mathfrak{F}_0\| \leq \|\partial_{y_1} \mathfrak{F}\|$  we obtain

$$\partial_{y_1} \|\mathfrak{F} - \mathfrak{F}_0\| \leq M(\|\mathfrak{F} - \mathfrak{F}_0\| + \|\mathfrak{F}_0\|),$$

and hence  $\|\mathfrak{F} - \mathfrak{F}_0\| \leq c(M)\|\mathfrak{F}_0\|a$ . Combining this result with (4.10) we finally arrive at

$$(4.11) \quad \|\mathfrak{F}(y) - \mathfrak{F}(0)\| \leq ca.$$

From this and the implicit function theorem we conclude that there is a positive constant  $a$ , depending only on  $M$  and  $\Sigma$ , such that the mapping  $x = \mathbf{x}(y)$  takes diffeomorphically the cube  $Q_a$  onto some neighborhood of the point  $P$ , and satisfy inequalities (4.2).

Let us turn to the proof of **(P2)**. We begin with the observation that the manifold  $\mathbf{x}(\Sigma \cap \mathcal{O}_P)$  is defined by the equation

$$\Phi_0(y) := x_3(y) - F(x_1(y), x_2(y)) = 0, \quad y \in Q_a.$$

Let us show that  $\Phi_0$  is strictly monotone in  $y_3$  and has the opposite signs on the faces  $y_3 = \pm a$ . To this end note that the formula for  $\mathfrak{F}(0)$  along with (4.11) implies the estimates

$$|\partial_{y_3} x_3(y) - 1| + |\partial_{y_3} x_1(y)| + |\partial_{y_3} x_2(y)| \leq ca \text{ in } Q_{a_1}.$$

Thus we get

$$(4.12) \quad 1 - ca \leq \partial_{y_3} \Phi_0(y) = \partial_{y_3} x_3(y) - \partial_{x_i} F(x_1, x_2) \partial_{y_3} x_i(y) \leq 1 + ca.$$

On the other hand, by (4.11), we have the inequality  $|x_3(y)| \leq ca|y|$ , which along with (4.2) yields the estimate

$$(4.13) \quad |\Phi_0(y)| = |x_3(y)| + |F(x(y))| \leq ca|y| + KC_M|y|^2 \leq ca^2 \text{ for } y_3 = 0.$$

Combining (4.12) and (4.13), we conclude that there exists a positive  $a$  depending only on  $M$  and  $\Sigma$ , such that the inequalities

$$(4.14) \quad 1/2 \leq \partial_{y_3} \Phi_0(y) \leq 2, \quad \pm \Phi_0(y_1, y_2, \pm a) > 0,$$

hold true for all  $y \in Q_a$ . Therefore, the equation  $\Phi_0(y) = 0$  has the unique solution  $y_3 = \Phi(y_1, y_2)$  in the cube  $Q_a$ , this solution vanishes for  $y_1 = y_2 = 0$ . By the implicit function theorem, the function  $\Phi$  belongs to the class  $C^1([-a, a]^2)$ .

It remains to prove that  $\Phi$  admits the both-side estimates (4.3). Note that inequality (4.2) implies the estimate  $|\mathbf{u}(\mathbf{x}(y)) - U\mathbf{e}_1| \leq M|\mathbf{x}(y)| \leq MC_M a$ . Therefore, we can choose  $a = a(M, \Sigma)$  sufficiently small, such that

$$2U/3 \leq u_1 \leq 4U/3, \quad C_\Gamma|u_2| \leq U/3.$$

Recall that  $x_1(y) - \Upsilon(x_2(y))$  vanishes at the plane  $y_1 = 0$  and

$$\partial_{y_1} [x_1(y) - \Upsilon(x_2(y))] = u_1(y) - \Upsilon'(x_2(y))u_2(y).$$

Since  $|\Upsilon'| \leq C_\Gamma$ , we obtain from this that

$$(4.15) \quad |y_1|U/3 \leq |x_1(y) - \Upsilon(x_2(y))| \leq |y_1|5U/3 \text{ for } y \in Q_a.$$

Equations (4.1) implies the identity

$$\partial_{y_1} \Phi_0(y) \equiv \nabla F_0(\mathbf{x}(y)) \cdot \mathbf{u}(x(y)) = \nabla F_0(\mathbf{x}(y)) \cdot \mathbf{U}(x(y)) \text{ for } \Phi_0(y) = 0.$$

Combining this result with (1.16) and (4.15), we finally obtain the inequality,

$$(4.16) \quad |y_1|N^-U/3 \leq |\partial_{y_1} \Phi_0(y)| \leq |y_1|N^+U/5,$$

which along with estimate (4.14) and the identity  $\partial_{y_1} \Phi = -\partial_{y_1} \Phi_0 (\partial_{y_3} \Phi_0)^{-1}$  yields (4.4). Since the term  $x_1(y) - \Upsilon(x_2(y))$  is positive for positive  $y_1$ , the function  $\Phi$  is increasing in  $y_1$  for  $y_1 > 0$  and is decreasing for  $y_1 < 0$ , which implies the existence of the functions  $a^\pm$ . Next, the identities  $\partial_{y_i} a^\pm = -\partial_{y_i} \Phi_0 / \partial_{y_1} \Phi_0$ ,  $i = 2, 3$ , and estimate (4.16) yield the inequality

$$|\partial_{y_i} a^\pm(y)| \leq c|y_1|^{-1}.$$

On the other hand, for  $y_1 = a^\pm(y_2, y_3)$ , we have  $y_3 = |\Phi(y_1, y_2)| \geq cy_1^2$  and hence  $|y_1|^{-1} \leq cy_3^{-1/2}$ , which implies the first estimates in (4.5). The second estimate is obvious.

In order to prove inclusions (4.7) note that  $\Phi(-a, y_2) \geq a^2 C^-$  and hence  $B_0(r) \cap \{y_3 > \Phi\} \subset G_a \subset Q_a$  for  $r = a^2 C^-$ . But estimate (4.2) implies that  $B_P(\rho_c) \subset \mathbf{x}(B_0(r))$  for  $\rho_c = rC_M^{-1}$ , which yields the first inclusion in (4.7). It remains to note that the second is a consequence of (4.2) and the lemma follows.  $\square$

The next lemma constitutes the existence of the normal coordinates in the vicinity of points of the inlet  $\Sigma_{\text{in}}$ .

**Lemma 4.2.** *Let vector fields  $\mathbf{u}$  and  $\mathbf{U}$  meet all requirements of Lemma 4.1 and  $U_n = -\mathbf{U}(P) \cdot \mathbf{n} > N > 0$ . Then there is  $b > 0$ , depending only on  $N$ ,  $\Sigma$  and  $M = \|\mathbf{u}\|_{C^1(\Omega)}$ , with the following properties. There exists a mapping  $y \rightarrow \mathbf{x}(y)$ , which takes diffeomorphically the cube  $Q_b = [-b, b]^3$  onto a neighborhood  $\mathcal{O}_P$  of  $P$  and satisfies the equations*

$$(4.17) \quad \partial_{y_3} \mathbf{x}(y) = \mathbf{u}(\mathbf{x}(y)) \text{ in } Q_b, \quad \mathbf{x}(y_1, y_2, 0) \in \Sigma \cap \mathcal{O}_P \text{ for } |y_2| \leq a,$$

and the inequalities

$$(4.18) \quad \|\mathbf{x}\|_{C^1(Q_b)} + \|\mathbf{x}^{-1}\|_{C^1(\mathcal{O}_P)} \leq C_{M,N} \quad |\mathbf{x}(y)| \leq C_M |y|,$$

where  $C_{M,N} = 3(1 + N^{-1})(M^2 + 2)^{1/2}$ . The inclusions

$$(4.19) \quad B_P(\rho_i) \cap \Omega \subset \mathbf{x}(Q_b \cap \{y_3 > 0\}) \subset B_P(R_i) \cap \Omega,$$

hold true for  $\rho_i = C_{M,N}^{-1}b$  and  $R_i = C_{M,N}b$ .

*Proof.* The proof simulates the proof of the Lemma 4.1. Choose the local Cartesian coordinates  $(x_1, x_2, x_3)$  centered at  $P$  such that in new coordinates  $\mathbf{n} = \mathbf{e}_3$ . By the smoothness of  $\Sigma$ , there is a neighborhood  $\mathcal{O} = [-k, k]^2 \times [-t, t]$  such that the manifold  $\Sigma \cap \mathcal{O}$  is defined by the equation

$$x_3 = F(x_1, x_2), \quad F(0, 0) = 0, \quad |\nabla F(x_1, x_2)| \leq K(|x_1| + |x_2|).$$

The constants  $k$ ,  $t$  and  $K$  depend only on  $\Sigma$ . Let us consider the initial value problem

$$(4.20) \quad \partial_{y_3} \mathbf{x} = \mathbf{u}(\mathbf{x}(y)) \text{ in } Q_a, \quad \mathbf{x}|_{y_3=0} = (y_1, y_2, F(y_1, y_2)).$$

Without loss of generality we can assume that  $0 < b < k < 1$ . It follows from (H1) that for any such  $b$  problem (4.20) has the unique solution of class  $C^1(Q_b)$ . Next, note that for  $y_3 = 0$  we have

$$(4.21) \quad |\mathbf{x}(y)| \leq (K+1)|y|, \quad |\mathbf{u}(\mathbf{x}(y)) - \mathbf{u}(0)| \leq M(K+1)|y|.$$

Denote by  $\mathfrak{F}(y) = D_y \mathbf{x}(y)$ . The calculations show that

$$\mathfrak{F}_0 := \mathfrak{F}(y)|_{y_3=0} = \begin{pmatrix} 1 & 0 & u_1 \\ 0 & 1 & u_2 \\ 0 & 0 & u_3 \end{pmatrix}, \quad \mathfrak{F}(0) = \begin{pmatrix} 1 & 0 & u_1(P) \\ 0 & 1 & u_2(P) \\ 0 & 0 & U_n \end{pmatrix},$$

which along with (4.21) implies

$$(4.22) \quad \|\mathfrak{F}(0)^{\pm 1}\| \leq C_{M,N}/3, \quad \|\mathfrak{F}_0(y) - \mathfrak{F}(0)\| \leq cb.$$

Next, differentiation of (4.20) with respect to  $y$  leads to the equation

$$\partial_{y_1} \mathfrak{F} = D_y \mathbf{u}(\mathbf{x}) \mathfrak{F}, \quad \mathfrak{F}|_{y_3=0} = \mathfrak{F}_0.$$

Arguing as in the proof of Lemma 4.1 we obtain  $\|\mathfrak{F} - \mathfrak{F}_0\| \leq c(M)\|\mathfrak{F}_0\|b$ . Combining this result with (4.22) we finally arrive at  $\|\mathfrak{F}(y) - \mathfrak{F}(0)\| \leq cb$ . From this and the implicit function theorem we conclude that there is positive  $b$ , depending only on  $M$  and  $\Sigma$ , such that the mapping  $x = \mathbf{x}(y)$  takes diffeomorphically the cube  $Q_b$  onto some neighborhood of the point  $P$ , and satisfies inequalities (4.18). Inclusions (4.19) easily follows from (4.18).  $\square$

*Model equation.* Assume that the function  $\Phi : [-a, a]^2 \mapsto \mathbb{R}$  and the constant  $a > 0$  meet all requirements of Lemma 4.1. Recall that for each  $y$  satisfying the conditions  $\Phi(y_1, y_2) < y_3 < \Phi(-a, y_2)$  ( resp.  $\Phi(y_1, y_2) < y_3 < \Phi(a, y_2)$ ), equation  $y_3 = \Phi(y_1, y_2)$  has the solutions  $y_1 = a^-(y_2, y_3)$  ( resp.  $y_1 = a^+(y_2, y_3)$ ). The solutions vanish for  $y_1 = 0$  and satisfy the inequalities

$$(4.23) \quad -a < a^-(y_2, y_3) \leq 0 \leq a^+(y_2, y_3) \leq a, \quad \partial_{y_i} a^\pm \leq C^* y_3^{-1/2}, \quad i = 2, 3,$$

where  $C^*$  depends only on  $K, C_\Gamma$ , and  $\mathbf{U}$ . We assume that the functions  $a^\pm$  are extended on the rectangle  $[-a, a] \times [0, a]$  by the equalities  $a^\pm(y_2, y_3) = \pm a$  for  $y_3 > \Phi(\pm a, y_2)$ . It is clear that the extended functions satisfy (4.23) and

$$Q_a^\phi := \{y_3 > \Phi(y_1, y_2)\} = \{y : a^-(y_2, y_3) \leq y_1 \leq a^+(y_2, y_3)\}.$$

Let us consider the boundary value problem

$$(4.24) \quad \partial_{y_1} \varphi(y) + \sigma \varphi(y) = f(y) \text{ in } Q_a^\phi, \quad \varphi(y) = 0 \text{ for } y_1 = a^-(y_2, y_3).$$

**Lemma 4.3.** *Assume that*

$$(4.25) \quad 1/2 < s \leq 1, \text{ and } 1 < r < 3/(2s-1).$$

*Then for any  $f \in H^{s,r}(Q_a) \cap L^\infty(Q_a^\phi)$ , problem (4.24) has a unique solution satisfying the inequalities*

$$(4.26) \quad \begin{aligned} \|\varphi\|_{H^{s,r}(Q_a^\phi)} &\leq c(r, s) \left( a^{4/r-s} \|f\|_{L^\infty(Q_a^\phi)} + a^{1/r} \|f\|_{H^{s,r}(Q_a^\phi)} \right), \\ \|\varphi\|_{L^\infty(Q_a^\phi)} &\leq \sigma^{-1} \|f\|_{L^\infty(Q_a^\phi)}. \end{aligned}$$



*Proof.* It suffices to prove the lemma in the case of  $s < 1$ . For every  $y \in \mathbb{R}^3$ , we denote by  $Y = (y_2, y_3)$ ,  $Z = (z_2, z_3)$ . Obviously, we have

$$(4.27) \quad \varphi(y) = \int_{a^-(Y)}^{y_1} e^{\sigma(x_1 - y_1)} f(x_1, Y) dx_1 \text{ and } \sigma \|\varphi\|_{C(Q_a^\phi)} \leq \|f\|_{C(Q_a^\phi)}.$$

Therefore, it suffices to estimate the semi-norm  $|\varphi|_{s,r,Q_a^\phi}$ . Choose an arbitrary  $y, z \in Q_a^\phi$ . Without any loss of generality we can assume that  $a^-(Z) \leq a^-(Y)$ . The identity

$$\begin{aligned} \varphi(z) - \varphi(y) &= \varphi(z_1, Z) - \varphi(y_1, Z) + \int_{a^-(Z)}^{a^-(Y)} e^{\sigma(x_1 - y_1)} f(x_1, Z) dx_1 + \\ &\quad \int_{a^-(Y)}^{y_1} e^{\sigma(x_1 - y_1)} (f(x_1, Z) - f(x_1, Y)) dx_1 \end{aligned}$$

implies the estimate

$$|\varphi(z) - \varphi(y)| \leq \|f\|_{L^\infty(Q_a^\phi)} (2|y_1 - z_1| + |a^-(Y) - a^-(Z)|) + \int_{-a}^a |(f(x_1, Z) - f(x_1, Y))| dx_1,$$

which along with the inequality

$$\left( \int_{-a}^a |(f(x_1, Z) - f(x_1, Y))| dx_1 \right)^r \leq a^{r-1} \int_{-a}^a |(f(x_1, Z) - f(x_1, Y))|^r dx_1$$

leads to the estimate

$$(4.28) \quad |\varphi|_{s,r,Q_a^\phi}^r \leq 2\|f\|_{C(Q_a^\phi)}^r (I_1 + I_2) + a^{(r-1)} I_3.$$

Here we denote

$$\begin{aligned} I_1 &= \int_{Q_a \times Q_a} \frac{|y_1 - z_1|^r}{|x - y|^{3+rs}} dx dy, \quad I_2 = \int_{Q_a^\phi \times Q_a^\phi} \frac{|a^-(Y) - a^-(Z)|^r}{|x - y|^{3+rs}} dx dy, \\ I_3 &= \int_{Q_a \times Q_a} \int_{-a}^a \frac{|f(x_1, Y) - f(x_1, Z)|^r}{|x - y|^{3+rs}} dx dy dx_1. \end{aligned}$$

Let us estimate the terms  $I_j$ ,  $i = 1, 2, 3$ . We begin with the observation that

$$\begin{aligned} \int_{[-a,a]^2} \frac{dZ}{|x - y|^{3+rs}} &= \frac{1}{|y_1 - z_1|^{3+rs}} \int_{[-a,a]^2} \frac{dZ}{(|Y - Z|^2/|y_1 - z_1|^2 + 1)^{(3+rs)/2}} \leq \\ &\frac{1}{|y_1 - z_1|^{1+rs}} \int_{\mathbb{R}^2} \frac{dZ}{(|Z|^2 + 1)^{(3+rs)/2}} \leq \frac{c}{|y_1 - z_1|^{1+rs}}, \end{aligned}$$

and hence

$$\int_{[-a,a]^4} \frac{dY dZ}{|x - y|^{3+rs}} \leq \frac{ca^2}{|y_1 - z_1|^{1+rs}}.$$

From this we obtain

$$(4.29) \quad I_1 \leq ca^2 \int_{-a}^a \left( \int_{-a}^a |y_1 - z_1|^{r(1-s)-1} dz_1 \right) dy_1 \leq c(r, s) a^{3+r(1-s)}.$$

In order to estimate  $I_2$ , note that by Lemma 4.1,

$$|a^-(Y) - a^-(Z)| \leq c|Y - Z|^{1/2}, \quad |a^-(Y) - a^-(Z)| \leq c|Y - Z| \left( \frac{1}{\sqrt{y_3}} + \frac{1}{\sqrt{z_3}} \right).$$

Next, it follows from the assumptions of lemma that there is  $\lambda \in (0, 1)$  such that

$$(4.30) \quad \lambda < 3/r, \quad 0 < (1 + \lambda)/2 - s < 1/r.$$

Noting that

$$|a^-(Y) - a^-(Z)| \leq c|Y - Z|^{(1+\lambda)/2} ((y_3)^{-\lambda/2} + (z_3)^{-\lambda/2}),$$

we obtain

$$\begin{aligned} I_2 \leq c \int_{Q_a^\phi} (y_3)^{-r\lambda/2} \left( \int_{Q_a} \frac{|Y - Z|^{r(1+\lambda)/2}}{|x - y|^{3+rs}} dz \right) dy + c \int_{Q_a^\phi} (z_3)^{-r\lambda/2} \left( \int_{Q_a} \frac{|Y - Z|^{r(1+\lambda)/2}}{|x - y|^{3+rs}} dy \right) dz \leq \\ 2c \int_{Q_a^\phi} (y_3)^{-r\lambda/2} \left( \int_{Q_a} \frac{|Y - Z|^{r(1+\lambda)/2}}{|x - y|^{3+rs}} dz \right) dy. \end{aligned}$$

Next, inequalities (4.30) imply

$$\begin{aligned} \int_{Q_a} \frac{|Y - Z|^{r(1+\lambda)/2}}{|x - y|^{3+rs}} dz \leq \int_{-a}^a |y_1 - z_1|^{-3-rs} \left( \int_{\mathbb{R}^2} \frac{|Y - Z|^{r(1+\lambda)/2}}{(|Y - Z|^2 |y_1 - z_1|^{-2} + 1)^{(3+rs)/2}} dZ \right) \leq \\ \int_{-a}^a |y_1 - z_1|^{-rs-1+r(1+\lambda)/2} dz_1 \int_{\mathbb{R}^2} \frac{|Z|^{r(1+\lambda)/2}}{(1 + |Z|^2)^{(3+rs)/2}} dZ \leq c(r, s) \int_{-a}^a |y_1 - z_1|^{r(1+\lambda)/2-rs-1} dz_1 \leq c(r, s) \end{aligned}$$

From this and Lemma 4.1 we conclude that

$$\begin{aligned} (4.31) \quad I_2 \leq c(r, s) \int_{Q_a^\phi} (y_3)^{-r\lambda/2} dy \leq c(r, s) \int_{[-a, a]^2} \left( \int_{cy_1^2 \leq y_3 \leq a} (y_3)^{-r\lambda/2} dy_3 \right) dy_1 dy_2 \\ \leq ac(r, s) \int_{-a}^a |y_1|^{2-r\lambda} dy_1 \leq ac(r, s). \end{aligned}$$

The reminding part of the proof is based on the following proposition.

**Proposition 4.4.** *Let  $f \in H^{s,r}(Q_a)$ . Then  $f$  has an extension  $\bar{f}$  over  $\mathbb{R}^3$ , which vanishes outside the set  $Q_{3a}$  and satisfies*

$$(4.32) \quad \|\bar{f}\|_{H^{s,r}} \leq ca^{(3-rs)/r} \|f\|_{L^\infty(Q_a)} + |f|_{s,r,Q_a}.$$

*Proof.* Define an extension of  $f$  onto the slab  $[-3a, 3a] \times [-a, a]^2$  by the formulae

$$f(x^\pm) = f(x) \text{ for } x \in Q_a, \text{ where } x^\pm = (\pm(2a - x_1), x_2, x_3).$$

It easily follows from the definition of the semi-norm  $|\cdot|_{s,r,\Omega}$ , that

$$\|f\|_{H^{s,r}([-3a, 3a] \times [-a, a]^2)} \leq 3\|f\|_{L^r(Q_a)} + 6|f|_{s,r,Q_a} \leq 6\|f\|_{H^{s,r}(Q_a)}.$$

Proceeding in the same way, we can extend  $f$  onto the plate  $[-3a, 3a]^2 \times [-a, a]$  and next, over the cube  $Q_{3a}$ . Obviously, the extended function satisfies the inequalities

$$(4.33) \quad \|f\|_{H^{s,r}(Q_{3a})} \leq 216\|f\|_{H^{s,r}(Q_a)}, \quad \|f\|_{C(Q_{3a})} \leq \|f\|_{C(Q_a)}.$$

Next, choose  $\mu \in C^\infty(\mathbb{R}^3)$  such that  $0 \leq \mu \leq 1$ ,  $\mu = 1$  in  $Q_1$  and  $\mu = 0$  outside of  $Q_{22}$ . Set  $\bar{f} = f\mu_a$ , where  $\mu_a(x) = \mu(x/a)$ . Next, the interpolation inequality along with the estimate  $|\nabla \mu_a| \leq ca^{-1}$  implies

$$\|\mu_a\|_{H^{s,r}(\mathbb{R}^3)} \leq \|\mu_a\|_{L^r(\mathbb{R}^3)}^{1-s} \|\mu_a\|_{H^{1,r}(\mathbb{R}^3)}^s \leq ca^{3(1-s)/r} a^{(3-r)s/r} = ca^{(3-rs)/r}.$$

From this and obvious inequality  $\|\mu_a f\|_{H^{s,r}(\mathbb{R}^3)} \leq \|f\|_{L^\infty(Q_{3a})} \|\mu_a\|_{H^{s,r}(\mathbb{R}^3)} + \|f\|_{H^{s,r}(Q_{3a})}$  we conclude that

$$\|\mu_a f\|_{H^{s,r}(\mathbb{R}^3)} \leq ca^{(3-rs)/r} \|f\|_{L^\infty(Q_a)} + \|f\|_{H^{s,r}(Q_a)}.$$

Hence  $\bar{f} = \mu_a f$  satisfies (4.32), and the proposition follows.  $\square$

Let us turn to proof of the lemma. We have

$$\begin{aligned} I_3 &= \int_{Q_a \times Q_a - a}^a |Z-Y|^{-3-rs} (|z_1-y_1|^2 |Z-Y|^{-2} + 1)^{-(3+rs)/2} |f(x_1, Z) - f(x_1, Y)|^r dx dy dx_1 \leq \\ &ca \int_{\mathbb{R}} (|t|^2 + 1)^{-(3+rs)/2} dt \int_{[-a,a]^5} |Z-Y|^{-2-rs} |f(x_1, Z) - f(x_1, Y)|^r dX dY dx_1 = \\ &ca \int_{-a}^a |f(x_1, \cdot)|_{r,s,[-a,a]^2}^r dx_1, \end{aligned}$$

which yields

$$(4.34) \quad I_3 \leq ca \|f\|_{L^r(-a,a;H^{s,r}([-a,a]^2))}^r \leq ca \|\bar{f}\|_{L^r(\mathbb{R};H^{s,r}(\mathbb{R}^2))}^r.$$

Recall that for  $s = 0, 1$ , the embedding operator  $H^{s,r}(\mathbb{R}^3) \hookrightarrow L^r(\mathbb{R}; H^{s,r}(\mathbb{R}^2))$  is bounded. By virtue of Lemma B.1, this results holds true for all  $s \in [0, 1]$ , which along with Proposition 4.4 and inequality (4.34) implies

$$I_3 \leq ca^{4-rs} \|f\|_{L^\infty(Q_a)}^r + a(|f|_{s,r,Q_a})^r.$$

Combining this result with (4.29), (4.31), since  $3 + r(1-s) \geq 4 - rs$ , we finally obtain

$$\|f\|_{L^\infty(Q_a)}^r (I_1 + I_2) + I_3 \leq ca^{4-rs} \|f\|_{L^\infty(Q_a)}^r + ca |f|_{s,r,Q_a}^r.$$

Substituting this inequality into (4.28) gives (4.26) and the lemma follows.  $\square$

Let us consider the following boundary value problem

$$(4.35) \quad \partial_{y_3} \varphi(y) + \sigma \varphi(y) = f(y) \text{ in } [-a, a]^2 \times [0, a], \quad \varphi(y) = 0 \text{ for } y_3 = 0.$$

**Lemma 4.5.** *Problem (4.35) has a unique solution satisfying the inequality*

$$(4.36) \quad \|\varphi\|_{H^{s,r}(Q_a^\phi)} \leq c(r, s) (a^{4/r-s} \|f\|_{L^\infty(Q_a^\phi)} + a^{1/r} \|f\|_{H^{s,r}(Q_a^\phi)}).$$

*Proof.* The proof of Lemma 4.3 can be used also in this case.  $\square$

*Local existence results.* It follows from the conditions of Theorem 1.3 that the vector field  $\mathbf{u}$  and the manifold  $\Sigma$  satisfy all assumptions of Lemma 4.1. Therefore, there exist positive numbers  $a$ ,  $\rho_c$  and  $R_c$ , depending only on  $\Sigma$  and  $\|\mathbf{u}\|_{C^1(\Omega)}$ , such that for all  $P \in \Gamma$ , the canonical diffeomorphism  $\mathbf{x} : Q_a \mapsto \mathcal{O}_P$  is well-defined and

meet all requirements of Lemma 4.1. Fix an arbitrary point  $P \in \Gamma$  and consider the boundary value problem

$$(4.37) \quad \mathbf{u} \cdot \nabla \varphi + \sigma \varphi = f \text{ in } \mathcal{O}_P, \quad \varphi = 0 \text{ on } \Sigma_{\text{in}} \cap \mathcal{O}_P.$$

**Lemma 4.6.** *Suppose that the exponents  $s, r$ , satisfy condition (1.30). Then for any  $f \in C^1(\Omega)$ , problem (4.37) has the unique solution satisfying the inequalities*

$$(4.38) \quad |\varphi|_{s,r,B_P(\rho_c)} \leq c(\|f\|_{C(B_P(R_c))} + |f|_{s,r,B_P(R_c)}), \quad \|\varphi\|_{C(B_P(\rho_c))} \leq \sigma^{-1}\|f\|_{C(B_P(R_c))},$$

where the constant  $c$  depends only on  $\Sigma$ ,  $M$ ,  $\sigma$ ,  $s, r$ , and  $\rho_c$  is determined by Lemma 4.1.

*Proof.* We transform equation (4.38) using the normal coordinates  $(y_1, y_2, y_3)$  given by Lemma 4.1. Set  $\bar{\varphi}(y) = \varphi(\mathbf{x}(y))$  and  $\bar{f}(y) = f(\mathbf{x}(y))$ . Next note that equation (4.1) implies the identity  $\mathbf{u} \nabla_x \varphi = \partial_{y_1} \bar{\varphi}(y)$ . Therefore the function  $\bar{\varphi}(y)$  satisfies the following equation and boundary conditions

$$(4.39) \quad \partial_{y_1} \bar{\varphi} + \sigma \bar{\varphi} = \bar{f} \text{ in } Q_a \cap \{y_3 > \Phi\}, \quad \bar{\varphi} = 0 \text{ for } y_3 = \Phi(y_1, y_2), \quad y_1 < 0.$$

It follows from Lemma 4.3 that problem (4.39) has the unique solution  $\bar{\varphi} \in H^{s,r}(G_a)$  satisfying the inequality

$$(4.40) \quad |\bar{\varphi}|_{s,r,G_a} \leq c(\|\bar{f}\|_{C(Q_a)} + |\bar{f}|_{s,r,Q_a}), \quad \|\bar{\varphi}\|_{C(G_a)} \leq \sigma^{-1}\|f\|_{C(Q_a)},$$

where the domain  $G_a$  is defined by (4.6). It remains to note that, by estimate (4.2), the mappings  $\mathbf{x}^{\pm 1}$  are uniformly Lipschitz, which along with inclusions (4.7) implies the estimates

$$|\varphi|_{s,r,B_P(\rho_c)} \leq c|\bar{\varphi}|_{s,r,G_a}, \quad |\bar{f}|_{s,r,Q_a} \leq c|f|_{s,r,B_P(R_c)}$$

Combining these results with (4.40) we finally obtain (4.38) and the lemma follows.  $\square$

In order to formulate the similar result for interior points of inlet we introduce the set

$$(4.41) \quad \Sigma'_{\text{in}} = \{x \in \Sigma_{\text{in}} : \text{dist}(x, \Gamma) \geq \rho_c/3\},$$

where the constant  $\rho_c$  is given by Lemma 4.1. It is clear that

$$\inf_{P \in \Sigma'_{\text{in}}} \mathbf{U}(P) \cdot \mathbf{n}(P) \geq N > 0,$$

where the constant  $N$  depends only on  $M$  and  $\Sigma$ . It follows from Lemma 4.2 that there are positive numbers  $b$ ,  $\rho_i$ , and  $R_i$  such that for each  $P \in \Sigma'_{\text{in}}$ , the canonical diffeomorphism  $\mathbf{x} : Q_b \mapsto \mathcal{O}_P$  is well-defined and satisfies the hypotheses of Lemma 4.2. The following lemma gives the local existence and uniqueness of solutions to the boundary value problem

$$(4.42) \quad \mathbf{u} \cdot \nabla \varphi + \sigma \varphi = f \text{ in } \mathcal{O}_P, \quad \varphi = 0 \text{ on } \Sigma_{\text{in}} \cap \mathcal{O}_P.$$

**Lemma 4.7.** *Suppose that the exponents  $s, r$  satisfy condition (1.37). Then for any  $f \in C^1(\Omega)$ , and  $P \in \Sigma'_{\text{in}}$ , problem (4.37) has the unique solution satisfying the inequalities*

$$(4.43) \quad |\varphi|_{s,r,B_P(\rho_i)} \leq c(\|f\|_{C(B_P(R_i))} + |f|_{s,r,B_P(R_i)}),$$

$$\|\varphi\|_{C(B_P(R))} \leq \sigma^{-1}\|f\|_{C(B_P(R_i))}.$$

where  $c$  depends on  $\Sigma$ ,  $M$ ,  $\sigma$  and exponents  $s, r$ .

*Proof.* Using the normal coordinates given by Lemma 4.2 we rewrite equation (4.42) in the form.

$$\partial_{y_3} \bar{\varphi} + \sigma \bar{\varphi} = \bar{f} \text{ in } Q_b, \quad \bar{\varphi} = 0 \text{ for } y_3 = 0.$$

Applying Lemma 4.23 and arguing as in the proof of Lemma 4.6 we obtain (4.43).  $\square$

Existence of solutions near inlet. The next step is based on the well-known geometric lemma (see Ch.3 in [19]).

**Lemma 4.8.** *Suppose that a given set  $A \subset \mathbb{R}^d$  is covered by balls such that each point  $x \in A$  is the center of a certain ball  $B_x(r(x))$  of radius  $r(x)$ . If  $\sup r(x) < \infty$ , then from the system of the balls  $\{B_x(r(x))\}$  it is possible to select a countable system  $B_{x_k}(r(x_k))$  covering the entire set  $A$  and having multiplicity not greater than a certain number  $n(d)$  depending only on the dimension  $d$ .*

The following lemma gives the dependence of the multiplicity of radiuses of the covering balls.

**Lemma 4.9.** *Assume that a collection of balls  $B_{x_k}(r) \subset \mathbb{R}^3$  of constant radius  $r$  has the multiplicity  $n_r$ . Then the multiplicity of the collections of the balls  $B_{x_k}(R)$ ,  $r < R$ , is bounded by the constant  $27(R/r)^3 n_r$ .*

*Proof.* Let  $n_R$  be a multiplicity of the system  $\{B_{x_k}(R)\}$ . This means that at least  $n_R$  balls, say  $B_{x_1}(R), \dots, B_{x_{n_R}}(R)$ , have the common point  $P$ . In particular, we have  $B_{x_i}(r) \subset B_P(3R)$  for all  $i \leq n_R$ . Introduce the counting function  $\iota(x)$  for the collection of balls  $B_{x_i}(r)$ , defined by

$$\iota(x) = \text{card}\{i : x \in B_{x_i}(r), 1 \leq i \leq n_r\}.$$

Note that  $\iota(x) \leq n_r$ . We have

$$\frac{4\pi}{3} n_R r^3 = \sum_{i=1}^{n_R} \text{meas } B_{x_i}(r) = \int_{\cup_i B_{x_i}(r)} \iota(x) dx \leq n_r \int_{\cup_i B_{x_i}(r)} dx \leq \frac{4\pi}{3} (3R)^3 n_r,$$

and the lemma follows.  $\square$

We are now in a position to prove the local existence and uniqueness of solution for the first boundary value problem for the transport equation in the neighborhood of the inlet. Let  $\Omega_t$  be the  $t$ -neighborhood of the set  $\Sigma_{\text{in}}$ ,

$$\Omega_t = \{x \in \Omega : \text{dist}(x, \Sigma_{\text{in}}) < t\},$$

**Lemma 4.10.** *Let  $t = \min\{\rho_c/2, \rho_i/2\}$  and  $T = \max\{R_c, R_i\}$ , where the constants  $\rho_\alpha, R_\alpha$  are defined by Lemmas 4.1 and 4.2. Then there exists a constant  $C$  depending only on  $M, \Sigma$  and  $\sigma$ , such that for any  $f \in C^1(\Omega)$ , the boundary value problem*

$$(4.44) \quad \mathbf{u} \cdot \nabla \varphi + \sigma \varphi = f \text{ in } \Omega_t, \quad \varphi = 0 \text{ on } \Sigma_{\text{in}}$$

*has the unique solution satisfying the inequalities*

$$(4.45) \quad |\varphi|_{s,r,\Omega_t} \leq C(\|f\|_{C(\Omega_T)} + |f|_{s,r,\Omega_T}), \quad \|\varphi\|_{C(\Omega_t)} \leq \sigma^{-1} \|f\|_{C(\Omega_T)}.$$

*Proof.* It follows from Lemma 4.8 that there is a covering of the characteristic manifold  $\Gamma$  by the finite collection of balls  $B_{P_i}(\rho_c/4)$ ,  $1 \leq i \leq m$ ,  $P_i \in \Gamma$ , of the multiplicity  $n$ . The cardinality  $m$  of this collection does not exceed  $4n(\rho_c)^{-1}L$ , where  $L$  is the length of  $\Gamma$ . Obviously, the balls  $B_{P_i}(\rho_c)$  cover the set

$$\mathcal{V}_\Gamma = \{x \in \Omega : \text{dist}(x, \Gamma) < \rho_c/2\}.$$

By virtue of Lemma 4.6, in each of such balls the solution to problem (4.44) satisfies inequalities (4.38), which leads to the estimate

$$(4.46) \quad |\varphi|_{s,r,\mathcal{V}_\Gamma}^r \leq \sum_i |\varphi|_{s,r,B_{P_i}(\rho_c)}^r \leq c \sum_i \|f\|_{C(B_{P_i}(R_c))}^r + c \sum_i |f|_{s,r,B_{P_i}(R_c)}^r,$$

where  $c$  depends only on  $M$ ,  $\Sigma$  and  $\sigma$ . By Lemma 4.9, the multiplicity of the system of balls  $B_{P_i}(R_c)$  is bounded from above by  $12^3(R_c/\rho_c)^d$ , which along with the inclusion  $\cup_i B_{P_i}(R_c) \subset \Omega_T$  yields

$$\sum_{i=1}^m |f|_{s,r,B_{P_i}(R_c)}^r \leq 12^3(R_c/\rho_c)^d |f|_{s,r,\Omega_T}^r.$$

Obviously we have

$$\sum_i \|f\|_{C(B_{P_i}(R_c))}^r \leq m \|f\|_{C(O_T)}^r \leq 4n(\rho_c)^{-1}L \|f\|_{C(O_T)}^r.$$

Combining these results with (4.46) we obtain the estimates for solution to problem (4.44) in the neighborhood of the characteristic manifold  $\Gamma$ ,

$$(4.47) \quad |\varphi|_{s,r,\mathcal{V}_\Gamma} \leq c \|f\|_{C(\Omega_T)} + c |f|_{s,r,\Omega_T}.$$

Our next task is to obtain the similar estimate in the neighborhood of the compact  $\Sigma'_{\text{in}} \subset \Sigma_{\text{in}}$ . To this end, we introduce the set

$$\mathcal{V}_{\text{in}} = \{x \in \Omega : \text{dist}(x, \Sigma'_{\text{in}}) < \rho_i/2\},$$

where  $\Sigma'_{\text{in}}$  is given by (4.41). By virtue of Lemma 4.8, there exists the finite collection of balls  $B_{P_k}(\rho_i/4)$ ,  $1 \leq k \leq m$ ,  $P_k \in \Sigma'_{\text{in}}$ , of the multiplicity  $n$  which covers  $\Sigma'_{\text{in}}$ . Obviously  $m \leq 16n(\rho_i)^{-2} \text{meas } \Sigma_{\text{in}}$ , and the balls  $B_{P_k}(\rho_i)$  cover the set  $\mathcal{V}_{\text{in}}$ . From this and Lemma 4.7 we conclude that

$$|\varphi|_{s,r,\mathcal{V}_{\text{in}}}^r \leq \sum_k |\varphi|_{s,r,B_{P_k}(\rho_i)}^r \leq c \sum_k \|f\|_{C(B_{P_k}(R_i))}^r + c \sum_k |f|_{s,r,B_{P_k}(R_i)}^r.$$

By virtue of Lemma 4.9, the multiplicity of the system of balls  $B_{P_i}(R_i)$  is not greater than  $12^3(R_i/\rho_i)^3$ , which yields

$$\sum_i |f|_{s,r,B_{P_i}(R_i)}^r \leq 12^3(R_i/\rho_i)^d |f|_{s,r,\Omega_T}^r$$

Obviously we have

$$\sum_k \|f\|_{C(B_{P_k}(R_i))}^r \leq m \|f\|_{C(\Omega_T)}^r \leq 16n(\rho_k)^{-2} \text{meas } \Sigma_{\text{in}} \|f\|_{C(\Omega_T)}^r.$$

Thus we get

$$(4.48) \quad |\varphi|_{s,r,\mathcal{V}_\Gamma} \leq c \|f\|_{C(O_T)} + c |f|_{s,r,\Omega_T}.$$

Since  $\mathcal{V}_\Gamma$  and  $\mathcal{V}_{\text{in}}$  cover  $\Omega_t$ , this inequality along with inequalities (4.47) yields (4.45), and the lemma follows.  $\square$

*Partition of unity.* Let us turn to the analysis of general problem

$$(4.49) \quad \mathcal{L}\varphi := \mathbf{u} \cdot \nabla \varphi + \sigma \varphi = f \text{ in } \Omega, \quad \varphi = 0 \text{ on } \Sigma_{\text{in}}.$$

The next step is based on the theory of partial differential equations with nonnegative characteristic form. The following lemma is a particular case of general results of Oleinik and Radkevich, we refer to Theorems 1.5.1 and 1.6.2 in [29].

**Lemma 4.11.** *Assume that  $\Omega$  is a bounded domain of the class  $C^2$ , the vector field  $\mathbf{u}$  belongs to the class  $C^1(\Omega)$ , and  $\sigma - \operatorname{div} \mathbf{u}(x) > \delta > 0$ . Then for any  $f \in L^\infty(\Omega)$ , problem (1.27) has a unique solution such that  $\|\varphi\|_{L^\infty(\Omega)} \leq \delta^{-1} \|f\|_{L^\infty}$ . Moreover, this solution is continuous in the interior points of  $\Sigma_{\text{in}}$  and vanishes on  $\Sigma_{\text{in}}$ . If, in addition,  $\Gamma = \operatorname{cl}(\Sigma_{\text{out}} \cap \Sigma_0) \cap \operatorname{cl} \Sigma_{\text{in}}$  is a smooth one dimensional manifold, then a bounded generalized solution to problem (4.49) is unique.*

The question of smoothness of solutions to boundary value problems for transport equations is more complicated. All known results [16], [29] related to the case of  $\Gamma = \emptyset$ . The following lemma is a consequence of Theorem 1.8.1 in the monograph [29].

**Lemma 4.12.** *Assume that  $\Omega$  is a bounded domain of the class  $C^2$  and  $\Sigma_{\text{out}} = \emptyset$ . Furthermore, let the following conditions hold.*

- 1) *The vector field  $\mathbf{u}$  and the function  $f$  belong to the class  $C^1(\mathbb{R}^3)$ .*
- 2) *There is  $\Omega' \ni \Omega$  such that the inequality*

$$\sigma - \sup_{\Omega'} \left\{ |\operatorname{div} \mathbf{u}| - \frac{1}{2} \sup_i \sum_{j \neq i} \left| \frac{\partial u_i}{\partial x_j} \right| - \frac{1}{2} \sup_j \left| \sum_{i \neq j} \frac{\partial u_j}{\partial x_i} \right| \right\} > 0.$$

*is fulfilled. Then a weak solution to problem (1.27) satisfies the Lipschitz condition in  $\bar{\Omega}$ .*

Using these results we can construct a strong solution to problem (1.27). Recall that by Lemma 1.39, for any  $f \in C^1(\Omega)$ , problem (4.49) has the unique strong solution defined in neighborhood  $\Omega_t$  of the inlet  $\Sigma_{\text{in}}$ . On the other hand, Lemma 4.11 guarantees the existence and uniqueness of bounded weak solution to problem (4.49). The following lemma shows that both the solutions coincide in  $\Omega_t$ .

**Lemma 4.13.** *Under the assumptions of Theorem 1.5 and Lemma 1.39, each bounded generalized solution to problem (4.49) coincides in  $\Omega_t$  with the local solution  $\varphi_t$ .*

*Proof.* Let  $\varphi \in L^\infty(\Omega)$  be a weak solution to problem (4.49). Recall that each point  $P \in \Gamma$  has a canonical neighborhood  $\mathcal{O}_P := \mathbf{x}(Q_a)$ , where canonical diffeomorphism  $\mathbf{x} : Q_a \mapsto \mathcal{O}_P$  is defined by Lemma 4.1. Choose an arbitrary function  $\zeta \in C^1(\Omega)$  vanishing on  $\Sigma_{\text{in}}$  and outside of  $\mathcal{O}_P$  and set

$$\bar{\varphi}(y) = \varphi(\mathbf{x}(y)), \quad \bar{f}(y) = f(\mathbf{x}(y)), \quad \bar{\zeta}(y) = \zeta(\mathbf{x}(y)), \quad y \in Q_a \cap \{y_3 > \Phi\}.$$

By the definition of the weak solution to the transport equation we have

$$\int_{\mathcal{O}_P \cap \Omega} (\sigma \varphi \zeta - \varphi \operatorname{div}(\zeta \mathbf{u}) - f \zeta) dx = 0,$$

Direct calculations lead to the identity  $\operatorname{div}_x(\zeta \mathbf{u}) = \det \mathfrak{F}^{-1} \operatorname{div}_y(\bar{\zeta} \det \mathfrak{F} \mathfrak{F}^{-1} \bar{\mathbf{u}})$ , in which the notation  $\mathfrak{F}$  stands for the Jacobi matrix  $\mathfrak{F} = D_y \mathbf{x}(y)$ . On the other hand,

equation (4.1) implies the equality  $\mathfrak{F}^{-1}\bar{\mathbf{u}} = \mathbf{e}_1$ . From this we conclude that

$$\int_{Q_a \cap \{y_3 > \Phi\}} \left( (\det \mathfrak{F} \bar{\zeta}) (\sigma \bar{\varphi} - \bar{f}) - \bar{\varphi} \frac{\partial}{\partial y_1} (\det \mathfrak{F} \bar{\zeta}) \right) dy = 0.$$

Recall that, by Lemma 4.1,  $\partial_{y_1} \mathfrak{F}$  is continuous and  $\det \mathfrak{F}$  is strictly positive in the cube  $Q_a$ . Setting  $\xi = \det \mathfrak{F} \bar{\zeta}$  we conclude that the integral identity

$$\int_{Q_a \cap \{y_3 > \Phi\}} \left( \xi (\sigma \bar{\varphi} - \bar{f}) - \bar{\varphi} \frac{\partial \xi}{\partial y_1} \right) dy = 0.$$

holds true for all functions  $\xi \in C_0(Q_a)$  having continuous derivative  $\partial_{y_1} \xi \in C(Q_a)$  and vanishing for  $y_3 = \Phi(y_1, y_2), y_1 < 0$ . Since  $\bar{f}$  is continuously differentiable,  $\bar{\varphi}$  belongs to the class  $C_{loc}^1(Q_a \cap \{y_3 > \Phi\})$ , and satisfies equations (4.39). On the other hand,  $\bar{\varphi}_t$  also satisfies (4.39). Obviously, all solutions of problem (4.39) coincides in the domain  $G_a$  and hence  $\bar{\varphi}_t = \bar{\varphi}$  in this domain. Recalling that  $B_P(\rho_c) \subset \mathbf{x}(G_a)$  we obtain that  $\varphi_t = \varphi$  in the ball  $B_P(\rho_c)$ . The same arguments show that for any  $P \in \Sigma_{in}^\rho$ , the function  $\varphi_t$  is equal to  $\varphi$  in the ball  $B_P(\rho_i)$ . It remains to note that the balls  $B_P(\rho_c)$  and  $B_P(\rho_i)$  cover  $\Omega_t$  and the lemma follows.  $\square$

Furthermore, we split the weak solution  $\varphi \in L^\infty(\Omega)$  to problem (4.49) into two parts, namely the local solution  $\varphi_t$  and the remainder vanishing near inlet. To this end fix a function  $\Lambda \in C^\infty(\mathbb{R})$  such that

$$(4.50) \quad 0 \leq \Lambda' \leq 3, \quad \Lambda(u) = 0 \text{ for } u \leq 1 \text{ and } \Lambda(u) = 1 \text{ for } u \geq 3/2,$$

and introduce the one-parametric family of smooth functions

$$(4.51) \quad \chi_t(x) = \frac{1}{t^3} \int_{\mathbb{R}^3} \Theta\left(\frac{2(x-y)}{t}\right) \Lambda\left(\frac{\text{dist}(y, \Sigma_{in})}{t}\right) dy$$

where  $\Theta \in C^\infty(\mathbb{R}^3)$  is a standard mollifying kernel supported in the unit ball. It follows that

$$(4.52) \quad \begin{aligned} \chi_t(x) &= 0 \text{ for } \text{dist}(x, \Sigma_{in}) \leq t/2, \quad \chi_t(x) = 1 \text{ for } \text{dist}(x, \Sigma_{in}) \geq 2t, \\ |\partial^l \chi_t(x)| &\leq \varpi(l) t^{-l} \text{ for all } l \geq 0. \end{aligned}$$

Now fix a number  $t = t(\Sigma, M)$  satisfying all assumptions of Lemma 1.39 and set

$$(4.53) \quad \varphi(x) = (1 - \chi_{t/2}(x)) \varphi_t(x) + \phi(x).$$

By virtue of (4.52) and Lemma 4.13, the function  $\phi \in L^\infty(\Omega)$  vanishes in  $\Omega_{t/2}$  and satisfies in a weak sense to the equations

$$\mathbf{u} \nabla \phi + \sigma \phi = \chi_{t/2} f + \varphi_t \mathbf{u} \nabla \chi_{t/2} =: F \text{ in } \Omega, \phi = 0 \text{ on } \Sigma_{in}$$

Next introduce new vector field  $\tilde{\mathbf{u}}(x) = \chi_{t/8}(x) \mathbf{u}(x)$ . It easy to see that  $\chi_{t/8} = 1$  on the support of  $\phi$  and hence the function  $\phi$  is also a weak solution to the modified transport equation

$$(4.54) \quad \tilde{\mathcal{L}} \phi := \tilde{\mathbf{u}} \nabla \phi + \sigma \phi = F \text{ in } \Omega.$$

The advantage of such approach is that the topology of integral lines of the modified vector field  $\tilde{\mathbf{u}}$  drastically differs from the topology of integral lines of  $\mathbf{u}$ . The corresponding inlet, outgoing set, and characteristic set have the other structure



and  $\tilde{\Sigma}_{\text{in}} = \emptyset$ . In particular, equation (4.54) does not require boundary conditions. Finally note that  $C^1$ -norm of the modified vector fields has the majorant

$$(4.55) \quad \|\tilde{\mathbf{u}}\|_{C^1(\Omega)} \leq M(1 + 16\varpi(1)t^{-1}),$$

where  $\varpi(1)$  is a constant from (4.52). The following lemma constitutes the existence and uniqueness of solutions to the modified equation.

**Lemma 4.14.** *Suppose that*

$$(4.56) \quad \sigma > \sigma^*(M, \Sigma) = 4M(1 + 16\varpi(1)t^{-1}) + 1, \quad M = \|\mathbf{u}\|_{C^1(\Omega)},$$

and  $0 \leq s \leq 1$ ,  $r > 1$ . Then for any  $F \in H^{s,r}(\Omega) \cap L^\infty(\Omega)$ , equation (4.54) has a unique weak solution  $\phi \in H^{s,r}(\Omega) \cap L^\infty(\Omega)$  such that

$$(4.57) \quad \|\phi\|_{H^{s,r}(\Omega)} \leq c\|F\|_{H^{s,r}(\Omega)}, \quad \|\phi\|_{L^\infty(\Omega)} \leq \sigma^{-1}\|F\|_{L^\infty(\Omega)},$$

where  $c$  depends only on  $r$ .

*Proof.* Without any loss of generality we can assume that  $F \in C^1(\Omega)$ . By virtue of (4.55) and (4.56), the vector field  $\tilde{\mathbf{u}}$  and  $\sigma$  meet all requirements of Lemma 4.12. Hence equation (4.54) has a unique solution  $\phi \in H^{1,\infty}(\Omega)$ . For  $i = 1, 2, 3$  and  $\tau > 0$ , define the finite difference operator

$$\delta_{i\tau}\phi = \frac{1}{\tau}(\phi(x + \tau\mathbf{e}_i) - \phi(x)).$$

It is easily to see that

$$(4.58) \quad \tilde{\mathbf{u}}\nabla\delta_{i\tau}\phi + \sigma\delta_{i\tau}\phi = \delta_{i\tau}F + \delta_{i\tau}\mathbf{u}\nabla\phi(\cdot + \tau\mathbf{e}_i) \text{ in } \Omega \cap (\Omega - \tau\mathbf{e}_i).$$

Next introduce the function  $\eta \in C^\infty(\mathbb{R})$  such that  $\eta' \geq 0$ ,  $\eta(u) = 0$  for  $u \leq 1$  and  $\eta(u) = 1$  for  $u \geq 1$ , and set  $\eta_h(x) = \eta(\text{dist}(x, \partial\Omega)/h)$ . Since  $\tilde{\Sigma}_{\text{in}} = \emptyset$ , the inequality

$$(4.59) \quad \limsup_{h \rightarrow 0} \int_{\Omega} g\tilde{\mathbf{u}} \cdot \nabla\eta_h(x) dx \leq 0$$

holds true for all nonnegative functions  $g \in L^\infty(\Omega)$ . Choosing  $h > \tau$ , multiplying both the sides of equation (4.58) by  $\eta_h|\delta_{i\tau}\phi|^{r-2}\delta_{i\tau}\phi$  and integrating the result over  $\Omega \cap (\Omega - \tau\mathbf{e}_i)$  we obtain

$$\begin{aligned} \int_{\Omega \cap (\Omega - \tau\mathbf{e}_i)} \eta_h |\delta_{i\tau}\phi|^r \left( \sigma - \frac{1}{r} \text{div } \tilde{\mathbf{u}} \right) dx - \int_{\Omega \cap (\Omega - \tau\mathbf{e}_i)} |\delta_{i\tau}\phi|^r \tilde{\mathbf{u}} \nabla \eta_h dx = \\ \int_{\Omega \cap (\Omega - \tau\mathbf{e}_i)} (\delta_{i\tau}F + \delta_{i\tau}\mathbf{u}\nabla\phi(\cdot + \tau\mathbf{e}_i)) \eta_h |\delta_{i\tau}\phi|^{r-2} \delta_{i\tau}\phi dx. \end{aligned}$$

Letting  $\tau \rightarrow 0$  and then  $h \rightarrow 0$  and using inequality (4.59) we obtain

$$(4.60) \quad \int_{\Omega} |\partial_{x_i}\phi|^r \left( \sigma - \frac{1}{r} \text{div } \tilde{\mathbf{u}} \right) dx \leq \int_{\Omega} (\partial_{x_i}F + \partial_{x_i}\mathbf{u}\nabla\phi) |\partial_{x_i}\phi|^{r-2} \partial_{x_i}\phi dx.$$

Next note that

$$\sum_i \partial_{x_i}\mathbf{u}\nabla\phi |\partial_{x_i}\phi|^{r-2} \partial_{x_i}\phi \leq 3\|\tilde{\mathbf{u}}\|_{C^1(\Omega)} \sum_i |\partial_{x_i}\phi|^r.$$

On the other hand, since  $1/r + 3 \leq 4$ , inequalities (4.55) and (4.56) imply

$$\sigma - \left(\frac{1}{r} + 3\right)\|\tilde{\mathbf{u}}\|_{C^1(\Omega)} \geq \sigma - 4M(1 + 16\varpi(1)t^{-1}) \geq 1.$$

From this we conclude that

$$\sum_i \int_{\Omega} |\partial_{x_i} \phi|^r dx \leq \sum_i \int_{\Omega} |\partial_{x_i} \phi|^{r-1} |\partial_{x_i} F| dx \leq \left( \sum_i \int_{\Omega} |\partial_{x_i} \phi|^r dx \right)^{1/r} \left( \sum_i \int_{\Omega} |\partial_{x_i} F|^r dx \right)^{1/r}$$

which leads to the estimate

$$(4.61) \quad \|\nabla \phi\|_{L^r(\Omega)} \leq c(r) \|\nabla f\|_{L^r(\Omega)}$$

Next multiplying both the sides of (4.54) by  $\phi \eta_h$  and integrating the result over  $\Omega$  we get the identity

$$\int_{\Omega} \left( \sigma - \frac{1}{r} \operatorname{div} \tilde{\mathbf{u}} \right) \eta_h |\phi|^r dx - \int_{\Omega} |\phi|^r \tilde{\mathbf{u}} \nabla \eta_h dx = \int_{\Omega} F \eta_h |\phi|^{r-2} \phi dx.$$

The passage  $h \rightarrow 0$  gives the inequality

$$\int_{\Omega} \left( \sigma - \frac{1}{r} \operatorname{div} \tilde{\mathbf{u}} \right) |\phi|^r dx \leq \int_{\Omega} |F| \eta_h |\phi|^{r-1} dx.$$

Recalling that  $\sigma - 1/r \operatorname{div} \tilde{\mathbf{u}} \geq 1$  we finally obtain

$$(4.62) \quad \|\phi\|_{L^r(\Omega)} \leq c(r) \|f\|_{L^r(\Omega)}.$$

Inequalities (4.61) and (4.62) imply estimate (4.57) for  $s = 0, 1$ . Hence the linear operator  $\tilde{\mathcal{L}}^{-1} : F \mapsto \phi$  is continuous in the Banach spaces  $H^{0,r}(\Omega)$  and  $H^{1,r}(\Omega)$  and its norm does not exceed  $c(r)$ . Recall that  $H^{s,r}(\Omega)$  is the interpolation space  $[L^r(\Omega), H^{1,r}(\Omega)]_{s,r}$ . From this and Lemma B.1 we conclude that inequality (4.57) is fulfilled for all  $s \in [0, 1]$ , which completes the proof.  $\square$

*Proof of Theorem 1.3.* We begin with the proof of the statement (i). Fix  $\sigma > \sigma^*$ , where the constant  $\sigma^*$  depends only on  $\Sigma$ ,  $\mathbf{U}$  and  $\|\mathbf{u}\|_{C^1(\Omega)}$ , and it is defined by (4.56). Without any loss of generality we can assume that  $f \in C^1(\Omega)$ . The existence and uniqueness of a weak bounded solution for  $\sigma > \sigma^*$ , follows from Lemma 4.11. Moreover, by virtue of Lemma 4.11, such a solution satisfies the second inequality in (1.32). Therefore, it suffices to prove estimate (1.32) for  $\|\varphi\|_{H^{s,r}(\Omega)}$ . Since  $H^{s,r}(\Omega) \cap L^\infty(\Omega)$  is the Banach algebra, representation (4.53) together with inequality (4.52) implies

$$(4.63) \quad \|\varphi\|_{H^{s,r}(\Omega)} \leq c(1+t^{-1})(\|\varphi_t\|_{H^{s,r}(\Omega_t)} + \|\varphi_t\|_{L^\infty(\Omega_t)}) + c\|\phi\|_{H^{s,r}(\Omega)}.$$

On the other hand, Lemma 4.14 along with (4.54) yields

$$\|\phi\|_{H^{s,r}(\Omega)} \leq c\|F\|_{H^{s,r}(\Omega)} \leq c\|\chi_{t/2} f\|_{H^{s,r}(\Omega)} + \|\varphi_t \mathbf{u} \nabla \chi_{t/2}\|_{H^{s,r}(\Omega)}.$$

The first terms in the right hand side is bounded,

$$\|\chi_{t/2} f\|_{H^{s,r}(\Omega)} \leq c(1+t^{-1})(\|f\|_{H^{s,r}(\Omega)} + \|\varphi_t\|_{L^\infty(\Omega)}).$$

In order to estimate the second term we note that, by virtue of (4.52),  $\|\mathbf{u} \nabla \chi_{t/2}\|_{C^1(\Omega)} \leq cM(1+t^{-2})$  which gives

$$\|\varphi_t \mathbf{u} \nabla \chi_{t/2}\|_{H^{s,r}(\Omega)} \leq cM(1+t^{-2})(\|\varphi_t\|_{H^{s,r}(\Omega_t)} + \|\varphi_t\|_{L^\infty(\Omega_t)}).$$

Substituting the obtained estimates into (4.63) we arrive at the inequality

$$\|\varphi\|_{H^{s,r}(\Omega)} \leq c(M+1)(1+t^{-2})(\|\varphi_t\|_{H^{s,r}(\Omega)} + \|\varphi_t\|_{L^\infty(\Omega_t)} + \|f\|_{H^{s,r}(\Omega_t)} + \|f\|_{L^\infty(\Omega)}),$$

which along with (4.45) leads to the estimate (1.32). In order to prove statement (ii) of Theorem 1.3 we note that the adjoint equation can be written in the form

$$-\mathbf{u} \nabla \varphi^* + \sigma \varphi^* = f + \varphi^* \operatorname{div} \mathbf{u}$$

Since

$$\|\operatorname{div} \mathbf{u} \varphi^*\|_{H^{s,r}(\Omega)} \leq c(\|\operatorname{div} \mathbf{u}\|_{H^{s,r}(\Omega)} + \|\operatorname{div} \mathbf{u}\|_{C(\Omega)}) \|\varphi^*\|_{H^{s,r}(\Omega)}$$

we have

$$\|\operatorname{div} \mathbf{u} \varphi^*\|_{H^{s,r}(\Omega)} + \|\operatorname{div} \mathbf{u} \varphi^*\|_{C(\Omega)} \leq \delta(\|\varphi^*\|_{H^{s,r}(\Omega)} + \|\varphi^*\|_{C(\Omega)}),$$

and the needed result follows from (i) and the contraction mapping principle.  $\square$

#### APPENDIX A. PROOF OF LEMMAS 1.6 AND 2.2

*Proof of Lemma 1.6.* Since  $\partial\Omega$  belongs to the class  $C^1$ , functions  $\varphi, \varsigma$  have the extensions  $\bar{\varphi}, \bar{\varsigma} \in H^{s,r}(\Omega) \cap H^{1,2}(\Omega)$ , such that  $\bar{\varphi}, \bar{\varsigma}$  are compactly supported in  $\mathbb{R}^d$  and

$$\|\bar{\varphi}\|_{H^{s,r}(\mathbb{R}^d)} \leq c\|\varphi\|_{H^{s,r}(\Omega)}, \quad \|\bar{\varsigma}\|_{H^{s,r}(\mathbb{R}^d)} \leq c\|\varsigma\|_{H^{s,r}(\Omega)}.$$

By virtue of Definition 1.1 and inequality (1.18), function  $\mathbf{w}$  has the extension by 0 outside  $\Omega$ , denoted by  $\bar{\mathbf{w}}$ , such that

$$\|\bar{\mathbf{w}}\|_{H^{1-s,r'}(\mathbb{R}^d)} \leq c\|\mathbf{w}\|_{\mathcal{H}_0^{1-s,r'}(\Omega)}.$$

Obviously we have

$$\mathfrak{B}(\mathbf{w}, \varphi, \varsigma) = - \int_{\mathbb{R}^d} \bar{\mathbf{w}} \cdot \nabla \bar{\varphi} \bar{\varsigma} dx,$$

The following multiplicative inequality is due to Mazja [23]. For all  $s > 0$ ,  $r > 1$  and  $rs < d$ ,

$$(5.1) \quad \|uv\|_{H^{s,r}(\mathbb{R}^d)} \leq c(r, s, d)(\|v\|_{H^{s,s/d}(\mathbb{R}^d)} + \|v\|_{L^\infty(\mathbb{R}^d)})\|u\|_{H^{s,r}(\mathbb{R}^d)}.$$

By virtue of (5.1), we have

$$\|\bar{\mathbf{w}} \bar{\varsigma}\|_{H^{1-s,r'}(\mathbb{R}^d)} \leq c\|\bar{\mathbf{w}}\|_{H^{1-s,r'}(\mathbb{R}^d)}(\|\bar{\varsigma}\|_{H^{1-s,d/(1-s)}(\mathbb{R}^d)} + \|\bar{\varsigma}\|_{L^\infty(\mathbb{R}^d)}).$$

On the other hand, since  $r^{-1} - (s - (1-s))/d \leq (1-s)/d$  for  $sr > d$ , embedding inequality (1.20) yields

$$\|\bar{\varsigma}\|_{H^{1-s,d/(1-s)}(\mathbb{R}^d)} \leq c\|\bar{\varsigma}\|_{H^{s,r}(\mathbb{R}^d)}, \quad \|\bar{\varsigma}\|_{L^\infty(\mathbb{R}^d)} \leq c\|\bar{\varsigma}\|_{H^{s,r}(\mathbb{R}^d)}.$$

Thus we get

$$\|\bar{\mathbf{w}} \bar{\varsigma}\|_{H^{1-s,r'}(\mathbb{R}^d)} \leq c\|\bar{\mathbf{w}}\|_{H^{1-s,r'}(\mathbb{R}^d)}\|\bar{\varsigma}\|_{H^{s,r}(\mathbb{R}^d)}.$$

It is well-known that elements of the fractional Sobolev spaces can be represented via Liouville potentials

$$\bar{\mathbf{w}} \bar{\varsigma} = (1 - \Delta)^{-(1-s)/2} w, \quad \bar{\varphi} = (1 - \Delta)^{-s/2} \phi,$$

with

$$\|w\|_{L_{r'}(\mathbb{R}^d)} \leq c\|\bar{\mathbf{w}} \bar{\varsigma}\|_{H^{1-s,r'}(\mathbb{R}^d)}, \quad \|\phi\|_{L_r(\mathbb{R}^d)} \leq c\|\bar{\varphi}\|_{H^{1-s,r'}(\mathbb{R}^d)}.$$

Thus we get

$$\mathfrak{B}(\mathbf{w}, \varphi, \varsigma) = - \int_{\mathbb{R}^d} (1 - \Delta)^{-(1-s)/2} w \cdot \nabla (1 - \Delta)^{-s/2} \phi dx = - \int_{\mathbb{R}^d} w \cdot \nabla (1 - \Delta)^{-1/2} \phi dx.$$

Since the Riesz operator  $(1 - \Delta)^{-1/2} \nabla$  is bounded in  $L^r(\mathbb{R}^d)$ , we conclude from this and the Hölder inequality that

$$|\mathfrak{B}(\mathbf{w}, \varphi, \varsigma)| \leq c \|w\|_{L^{r'}(\mathbb{R}^d)} \|\phi\|_{L^r(\mathbb{R}^d)} \leq c \|\mathbf{w}\|_{H^{1-s, r'}(\Omega)} \|\varphi\|_{H^{s, r}(\Omega)} \|\varsigma\|_{H^{s, r}(\Omega)},$$

and the lemma follows.  $\square$

*Proof of Lemma 2.2.* By virtue of (1.18), the extension  $\bar{\mathbf{w}}$  satisfies the inequalities

$$\|\bar{\mathbf{w}}\|_{H^{1-s, r'}(\mathbb{R}^3)} \leq c \|\mathbf{w}\|_{\mathcal{H}_0^{1-s, r'}(\Omega)}, \quad \|\bar{\mathbf{w}}\|_{H^{1, 2}(\mathbb{R}^3)} \leq c \|\mathbf{w}\|_{H^{1, 2}(\Omega)}.$$

On the other hand, the vector field  $\mathbf{h}$  has a compactly supported extension  $\bar{\mathbf{h}} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  such that  $\|\bar{\mathbf{h}}\|_{H^{1+s, r}(\mathbb{R}^3)} \leq c \|\mathbf{h}\|_{H^{1+s, r}(\Omega)}$ , but this extension does not vanish outside  $\Omega$ . Substituting the expression for  $\mathcal{A}$  into the formula for  $\mathfrak{A}$  and integrating by parts we conclude that  $\mathfrak{A}(\mathbf{w}, \mathbf{h})$  equals

$$\int_{\mathbb{R}^3} \mathbf{g}^{-1} \left( \nabla((\mathbf{N}^{-1} - \mathbf{I})\bar{\mathbf{w}}) : (\mathbf{N}\mathbf{N}^* \nabla(\mathbf{N}^{-1*} \bar{\mathbf{h}})) + \nabla \bar{\mathbf{w}} : (\mathbf{N}\mathbf{N}^* \nabla(\mathbf{N}^{-1*} \bar{\mathbf{h}}) - \mathbf{g} \nabla \bar{\mathbf{h}}) \right) dx$$

Since  $\|\mathbf{N}_1^{\pm 1} - \mathbf{I}\|_{C^2(\Omega)} \leq c\tau^2$ , we have

$$\begin{aligned} \|(\mathbf{N}^{-1} - \mathbf{I})\bar{\mathbf{w}}\|_{H^{1-s, r'}(\mathbb{R}^3)} &\leq c\tau^2 \|\bar{\mathbf{w}}\|_{H^{1-s, r'}(\mathbb{R}^3)}, \\ \|\mathbf{g}^{-1} \mathbf{N}\mathbf{N}^* \nabla(\mathbf{N}^{-1*} \bar{\mathbf{h}}) - \nabla \bar{\mathbf{h}}\|_{H^{s, r}(\mathbb{R}^3)} &\leq c\tau^2 \|\bar{\mathbf{h}}\|_{H^{1+s, r}(\mathbb{R}^3)}. \end{aligned}$$

It follows from this that the vector fields  $\mathbf{a}_0$ ,  $\mathbf{a}$  and the matrices  $\mathbf{V}_0$ ,  $\mathbf{V}$  defined by the relations

$$\begin{aligned} \bar{\mathbf{w}} &= (1 - \Delta)^{(s-1)/2} \mathbf{a}_0, \quad \mathbf{N}^{-1} \bar{\mathbf{w}} = (1 - \Delta)^{(s-1)/2} \mathbf{a}, \\ \nabla \bar{\mathbf{h}} &= (1 - \Delta)^{s/2} \mathbf{V}_0, \quad \mathbf{g}^{-1} \mathbf{N}\mathbf{N}^* \nabla(\mathbf{N}^{-1*} \bar{\mathbf{h}}) = (1 - \Delta)^{s/2} \mathbf{V}, \end{aligned}$$

satisfy the inequalities

$$(5.2) \quad \begin{aligned} \|\mathbf{a}_0\|_{L_{r'}(\mathbb{R}^3)} &\leq c \|\bar{\mathbf{w}}\|_{H^{1-s, r'}(\mathbb{R}^3)}, \quad \|\mathbf{a} - \mathbf{a}_0\|_{L_{r'}(\mathbb{R}^3)} \leq c\tau^2 \|\bar{\mathbf{w}}\|_{H^{1-s, r'}(\mathbb{R}^3)}, \\ \|\mathbf{V}_0\|_{L_r(\mathbb{R}^3)} &\leq c \|\bar{\mathbf{h}}\|_{H^{1+s, r}(\mathbb{R}^3)}, \quad \|\mathbf{V} - \mathbf{V}_0\|_{L_r(\mathbb{R}^3)} \leq c\tau^2 \|\bar{\mathbf{h}}\|_{H^{1+s, r}(\mathbb{R}^3)}. \end{aligned}$$

From this, the identity

$$\mathfrak{A}(\mathbf{w}, \mathbf{h}) = \int_{\mathbb{R}^3} \nabla(1 - \Delta)^{-1/2} (\mathbf{a} - \mathbf{a}_0) : \mathbf{V}_0 dx + \int_{\mathbb{R}^d} \nabla(1 - \Delta)^{-1/2} \mathbf{a}_0 : (\mathbf{V} - \mathbf{V}_0) dx$$

and the Hölder inequality we conclude that

$$|\mathfrak{A}_1(\mathbf{w}, \mathbf{h})| \leq c \|\mathbf{a} - \mathbf{a}_0\|_{L_{r'}(\mathbb{R}^3)} \|\mathbf{V}\|_{L_r(\mathbb{R}^3)} + \|\mathbf{a}_0\|_{L_{r'}(\mathbb{R}^3)} \|\mathbf{V} - \mathbf{V}_0\|_{L_r(\mathbb{R}^3)}.$$

Combining this result with (5.2) we obtain (2.25) and the lemma follows.

## APPENDIX B. INTERPOLATION

In this section we recall some results from the interpolation theory, see [6] for the proofs. Let  $A_0$  and  $A_1$  be Banach spaces. For  $t > 0$  introduce two non-negative functions  $K : A_0 + A_1 \mapsto \mathbb{R}$  and  $J : A_1 \cap A_1 \mapsto \mathbb{R}$  defined by

$$K(t, u, A_0, A_1) = \inf_{\substack{u = u_0 + u_1 \\ u_i \in A_i}} \|u_0\|_{A_0} + t \|u_1\|_{A_1}, \quad J(t, u, A_0, A_1) = \max\{\|u\|_{A_0}, t \|u\|_{A_1}\}.$$

For each  $s \in (0, 1)$ ,  $1 < r < \infty$ , the  $K$ -interpolation space  $[A_0, A_1]_{s,r,K}$  consists of all elements  $u \in A_0 + A_1$ , having the finite norm

$$(6.1) \quad \|u\|_{[A_0, A_1]_{s,r,K}} = \left( \int_0^\infty t^{-1-sr} K(t, u, A_0, A_1)^r dt \right)^{1/r}.$$

On the other hand,  $J$ -interpolation space  $[A_0, A_1]_{s,r,J}$  consists of all elements  $u \in A_0 + A_1$  which admit the representation

$$(6.2) \quad u = \int_0^\infty \frac{v(t)}{t} dt, \quad v(t) \in A_1 \cap A_0 \text{ for } t \in (0, \infty)$$

and have the finite norm

$$(6.3) \quad \|u\|_{[A_0, A_1]_{s,r,J}} = \inf_{v(t)} \left( \int_0^\infty t^{-1-sr} J(t, v(t), A_0, A_1)^r dt \right)^{1/r} < \infty,$$

where the infimum is taken over the set of all  $v(t)$  satisfying (6.2). The first main result of interpolation theory reads: For all  $s \in (0, 1)$  and  $r \in (1, \infty)$  the spaces  $[A_0, A_1]_{s,r,K}$  and  $[A_0, A_1]_{s,r,J}$  are isomorphic topologically and algebraically. Hence the introduced norms are equivalent, and we can omit indices  $J$  and  $K$ . The following simple properties of interpolation spaces directly follows from definitions.

1) If  $A_1 \subset A_0$  is dense in  $A_0$ , then  $[A_0, A_1]_{s,r} \subset A_0$  is dense in  $A_0$ . To show this fix an arbitrary  $u \in [A_0, A_1]_{s,r}$  and choose the  $v$  in representation (6.2) such that  $\|t^{-s}v\|_{L^r(0, \infty; dt/t)} < \infty$ . It is easily to see that  $u_n = \int_{n^{-1}}^n v(t)t^{-1}dt \in A_1$  and

$$\|u_n - u\|_{[A_0, A_1]_{s,r,J}}^r \leq \left( \int_0^{n^{-1}} + \int_n^\infty \right) t^{-1-sr} J(t, v(t), A_0, A_1)^r dt \rightarrow 0 \text{ as } n \rightarrow \infty.$$

2) If  $\tilde{A}_i$ ,  $i = 0, 1$ , are closed subspaces of  $A_i$ , then  $[\tilde{A}_0, \tilde{A}_1]_{s,r} \subset [A_0, A_1]_{s,r}$  and  $\|u\|_{[A_0, A_1]_{s,r}} \leq \|u\|_{[\tilde{A}_0, \tilde{A}_1]_{s,r}}$ .

One of the important results of the interpolation theory is the following representation for the interpolation of dual spaces. Let  $A_i$  be Banach spaces such that  $A_1 \cap A_0$  is dense in  $A_0 + A_1$ . Then the Banach spaces  $[(A_0)', (A_1)']_{s,r'}$  and  $([A_0, A_1]_{s,r})'$  are isomorphic topologically and algebraically. Hence the spaces can be identified with equivalent norms.

In particular, if  $A_1 \subset A_0$ ,  $A_0' \subset A_1'$  are dense in  $A_0$  and  $A_1'$ , then  $([A_0, A_1]_{s,r})'$  is the completion of  $A_0'$  in  $([A_0, A_1]_{s,r})'$ -norm.

The following lemma is the central result of the interpolation theory.

**Lemma B.1.** *Let  $A_i$ ,  $B_i$ ,  $i = 0, 1$ , be Banach spaces and let  $T : A_i \mapsto B_i$ , be a bounded linear operator. Then for all  $s \in (0, 1)$  and  $r \in (1, \infty)$ , the operator  $T : [A_0, A_1]_{s,r} \mapsto [B_0, B_1]_{s,r}$  is bounded and*

$$\|T\|_{\mathcal{L}([A_0, A_1]_{s,r}, [B_0, B_1]_{s,r})} \leq \|T\|_{\mathcal{L}(A_0, B_0)}^s \|T\|_{\mathcal{L}(A_1, B_1)}^{1-s}$$

Now we show that all basic properties of spaces  $\mathcal{H}_0^{s,r}$  determined by Definition 1.1 easy follows from mentioned results of the interpolation theory. Let  $\Omega$  be a bounded domain with a boundary of the class  $C^1$  or  $\Omega = \mathbb{R}^d$ . It is well-known that for all  $s \in (0, 1)$  and  $r \in (1, \infty)$ , the Sobolev space  $H^{s,r}(\Omega) = [L^r(\Omega), H^{1,r}(\Omega)]_{s,r}$ . Since  $\mathcal{H}^{0,r}(\Omega)$  and  $\mathcal{H}_0^{1,r}(\Omega)$  are closed subspaces of  $H^{0,r}(\mathbb{R}^d)$  and  $H^{1,r}(\mathbb{R}^d)$ , the interpolating space  $\mathcal{H}_0^{s,r}$  determined by Definition 1.1 satisfies inequality (1.18).

Next note that, by virtue of pairing (1.21), the space  $L^{r'}(\Omega)$  can be identified with  $(\mathcal{H}_0^{0,r})'$ , which is dense in  $\mathcal{H}^{-1,r}(\Omega) = (\mathcal{H}_0^{1,r}(\Omega))'$ . Therefore, the space  $(\mathcal{H}_0^{s,r})'$  is the completion of  $L^{r'}(\Omega)$  in the norm of  $(\mathcal{H}_0^{s,r}(\Omega))'$ , which is exactly equal to

the norm of  $\mathcal{H}^{-s,r'}(\Omega)$ . Hence  $(\mathcal{H}_0^{s,r}(\Omega))' = \mathcal{H}^{-s,r'}(\Omega)$  which leads to the duality principle (1.23).

*Proof of Lemma 1.2.* Finally we show that Lemma 1.2 is a straightforward consequence of classical results on solvability of the first boundary value problem for the Stokes equations. Note that, by virtue of Theorem 6.1 in [7], for any  $\mathbf{F} \in \mathcal{H}^{s-1,r}(\Omega)$  and  $G \in H^{s,r}(\Omega)$  with  $s = 0, 1$ , problem (1.25) has the unique solution  $\mathbf{v}, \pi$  satisfying inequality

$$\|\mathbf{v}\|_{H^{s+1,r}(\Omega)} + \|\pi\|_{H^{s,r}(\Omega)} \leq c(\Omega, r, s)(\|\mathbf{F}\|_{\mathcal{H}^{s-1,r}(\Omega)} + \|G\|_{H^{s,r}(\Omega)})$$

Thus the relation  $(F, G) \mapsto (\mathbf{v}, \pi)$  determines the linear operator  $T : \mathcal{H}^{s-1,r}(\Omega) \times H^{s,r}(\Omega) \mapsto H^{s+1,r}(\Omega) \times H^{s,r}(\Omega)$ . Therefore, Lemma 1.2 is a consequence of Lemma B.1.  $\square$

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